MTR08114 Robotics Rotation

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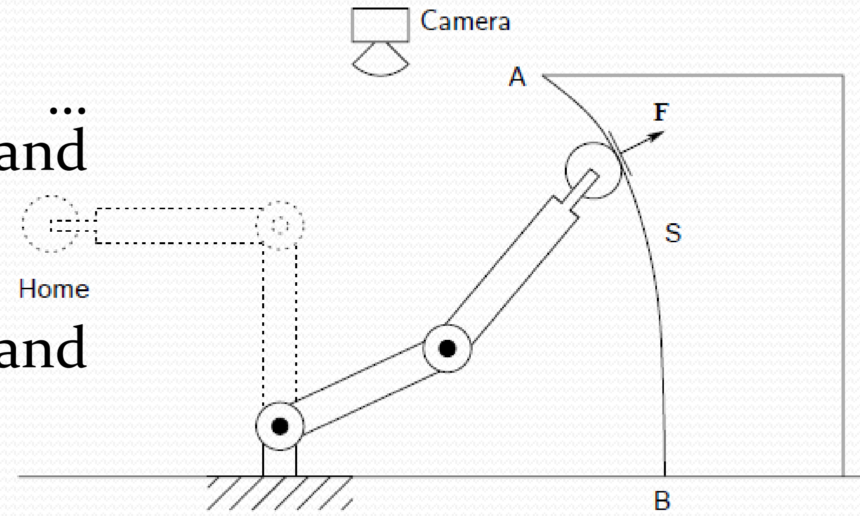
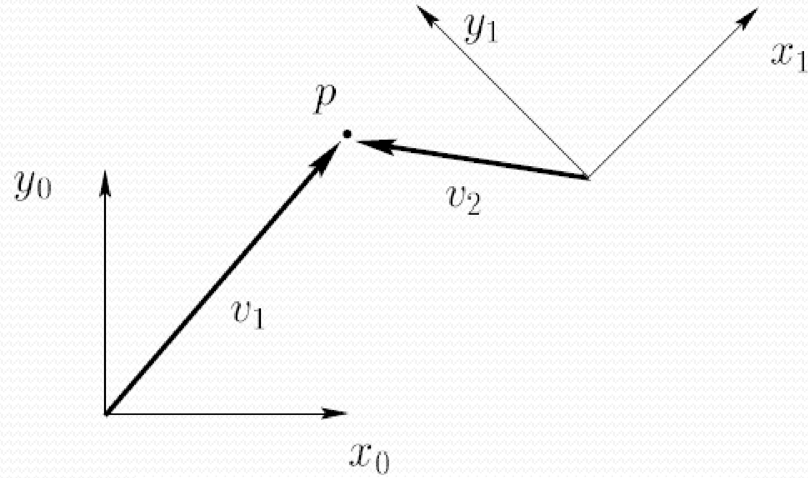
REMINDER 1: How To Do It

- Problem 1
  - Make our goal mathematical … From words to equations (and back!!)
  - This is called Representation and will be our first task
  - Examples:
    - How to represent all points of the robot as compactly as possible?
    - How to represent a rotation in space?
    - How can we move between different vintage points (reference frames)?
    - ……….



Representing Location

- *p* is a point
- p^o is the point *p* with respect to reference frame $x_o y_o$
- v_1 is a vector specifying the distance and direction of *p* with respect to origin o_o
- Free vectors have *no* position
- Points have *no* direction
- Vectors ≠ Points



$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad p^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$

$$o_1^0 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \quad o_0^1 = \begin{bmatrix} -10.6 \\ 3.5 \end{bmatrix}$$

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad v_1^1 = \begin{bmatrix} 7.77 \\ 0.8 \end{bmatrix}$$

$$v_2^0 = \begin{bmatrix} -5.1 \\ 1 \end{bmatrix}, \quad v_2^1 = \begin{bmatrix} -2.89 \\ 4.2 \end{bmatrix}$$

Coordinate Conventions

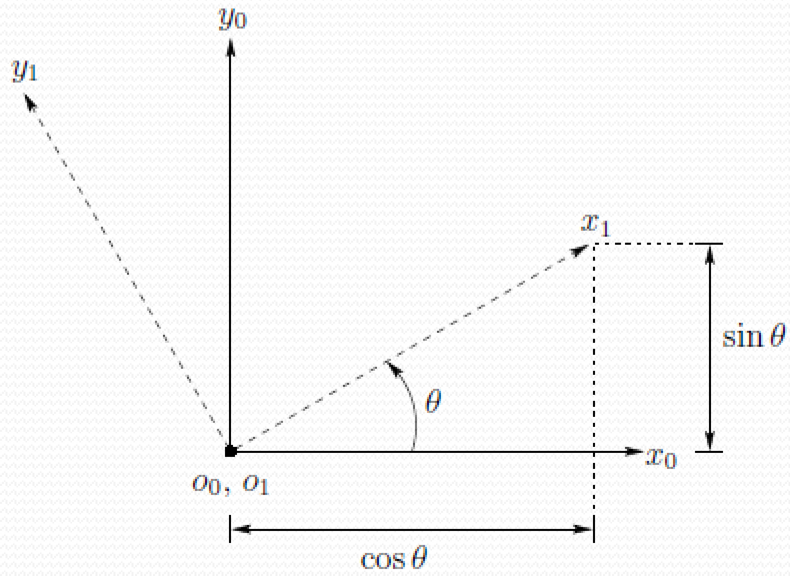
- To use algebraic operations all vectors **MUST** be in the same frame (has same super script)

Representing Rotation

- Solution 1
  - Just Specify Θ

- Disadvantage
  - Discontinuous
  - Does Not Scale

- Solution 2
  - Rotation Matrix



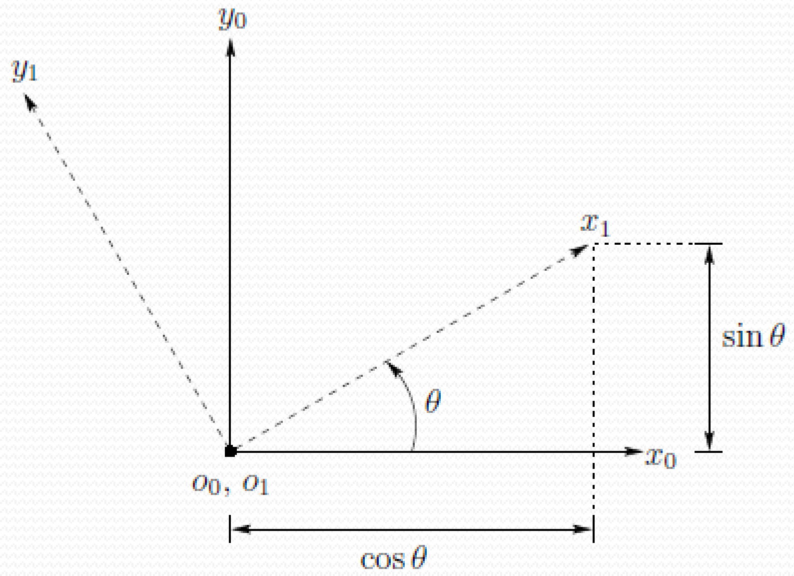
Rotation Matrix

$$R_1^0 = \left[x_1^0 \middle| y_1^0 \right]$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} x_1.x_0 & y_1.x_0 \\ x_1.y_0 & y_1.y_0 \end{bmatrix}$$

Direction Cosines



Properties of Rotation Matrices

SO(n) = Special Orthogonal Group of Order n

- $R \in SO(n)$
- $R^{-1} \in SO(n)$
- $R^{-1} = R^T$
- The columns (and therefore the rows) of R are mutually orthogonal
- Each column (and therefore each row) of R is a unit vector
- $\det R = 1$

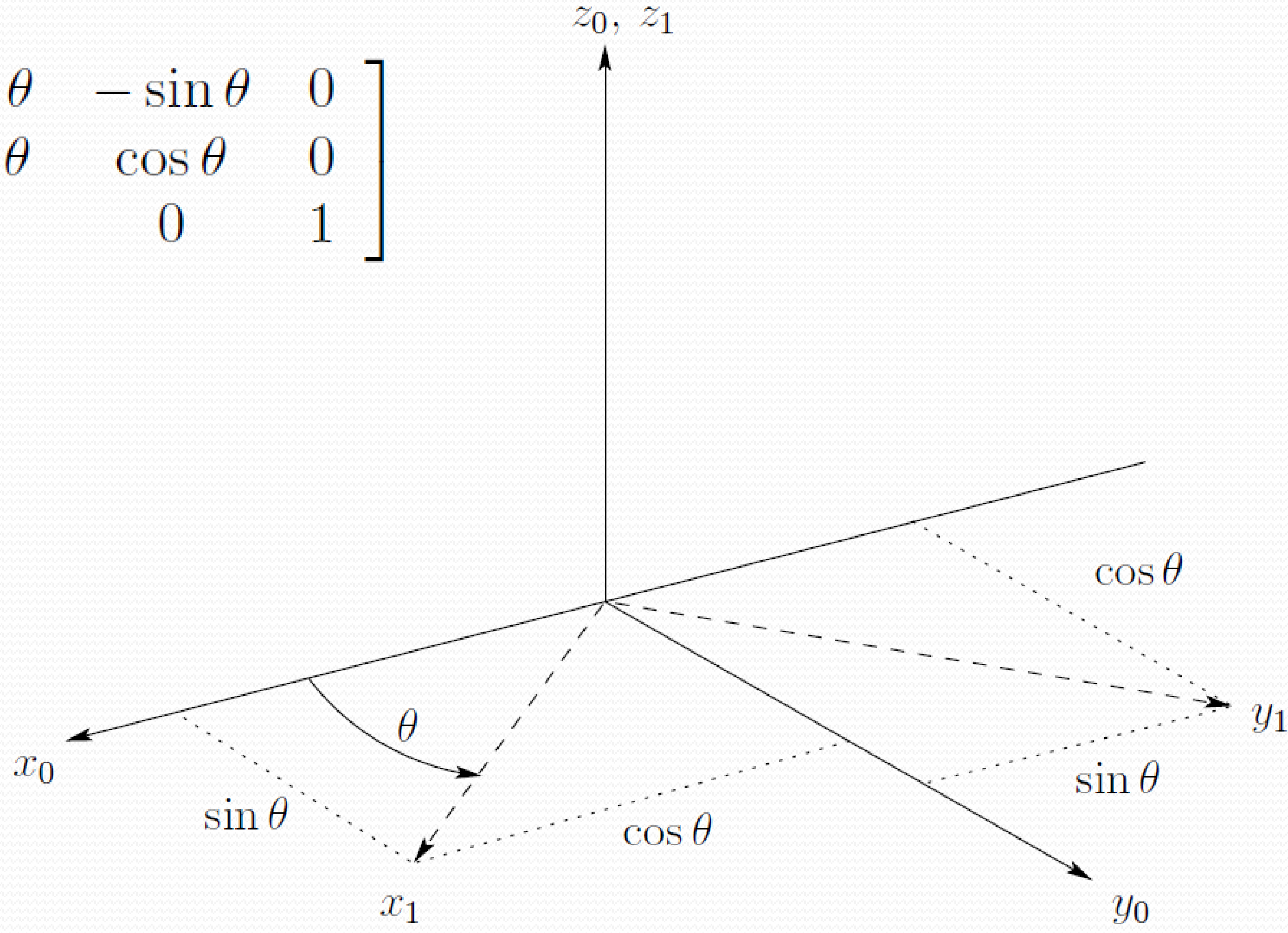
$$\begin{aligned} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \end{aligned}$$

Rotation in 3D

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Rotation Around Z axis

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation Around x and y

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

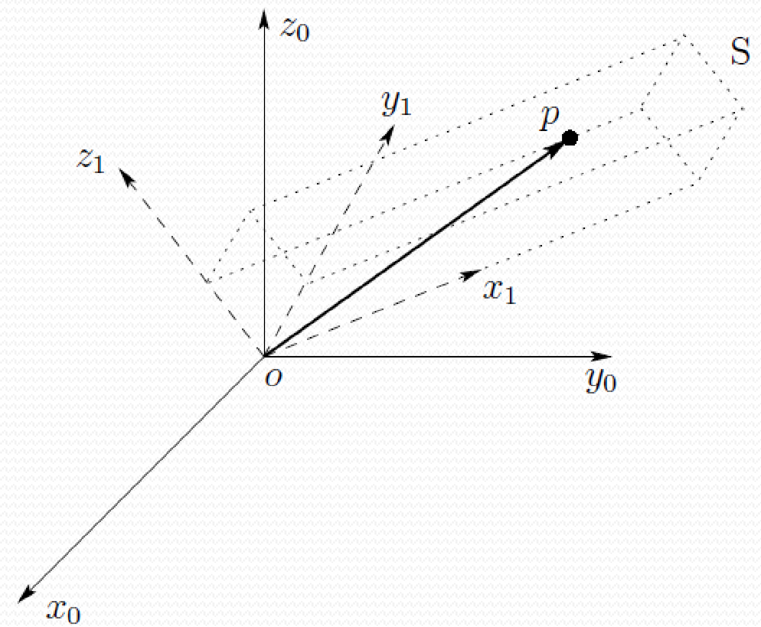
$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotational Transformation

$$p = ux_1 + vy_1 + wz_1$$

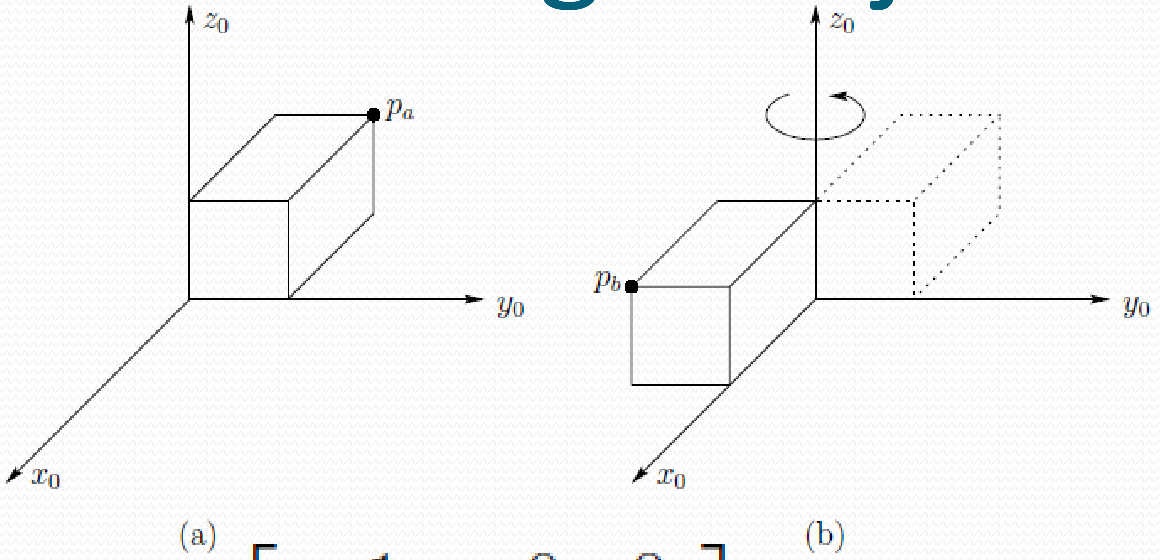
$$p^0 = \begin{bmatrix} p \cdot x_0 \\ p \cdot y_0 \\ p \cdot z_0 \end{bmatrix}$$

$$\begin{aligned} p^0 &= \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_0 \\ (ux_1 + vy_1 + wz_1) \cdot y_0 \\ (ux_1 + vy_1 + wz_1) \cdot z_0 \end{bmatrix} \\ &= \begin{bmatrix} ux_1 \cdot x_0 + vy_1 \cdot x_0 + wz_1 \cdot x_0 \\ ux_1 \cdot y_0 + vy_1 \cdot y_0 + wz_1 \cdot y_0 \\ ux_1 \cdot z_0 + vy_1 \cdot z_0 + wz_1 \cdot z_0 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \end{aligned}$$



$$p^0 = R_1^0 p^1$$

Rotation of rigid objects



$$R_1^0 = R_{z,\pi} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p_b^1 = p_a^0;$$

$$p_b^0 = R_{z,\pi} p_b^1$$

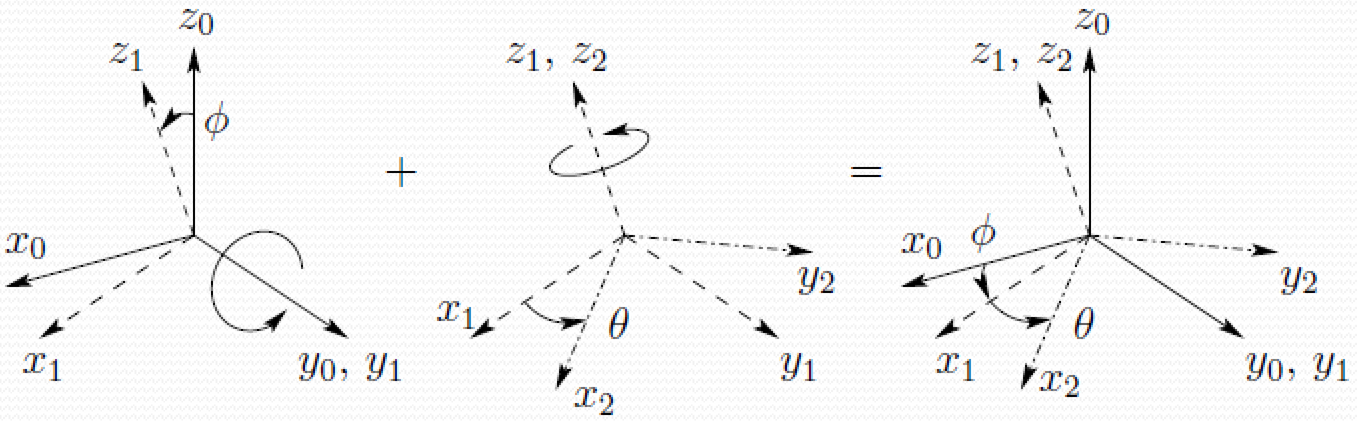
$$p_b^0 = R_{z,\pi} p_a^0$$

$$p_b^0 = R_{z,\pi} p_a^0$$

Meanings of R

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.
2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.
3. It is an operator taking a vector and rotating it to a new vector in the same coordinate system.

Composition of Rotations



$$
\begin{aligned}
p^0 &= R_1^0 p^1 \\
p^1 &= R_2^1 p^2 \\
p^0 &= R_2^0 p^2 \\
p^0 &= R_1^0 R_2^1 p^2
\end{aligned}
$$

$$R_2^0 = R_1^0 R_2^1$$

Example Composition

- Rotation around y axis followed by a rotation around the new z axis

$$
\begin{aligned}
R &= R_{y,\phi} R_{z,\theta} \\
&= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}
\end{aligned}
$$

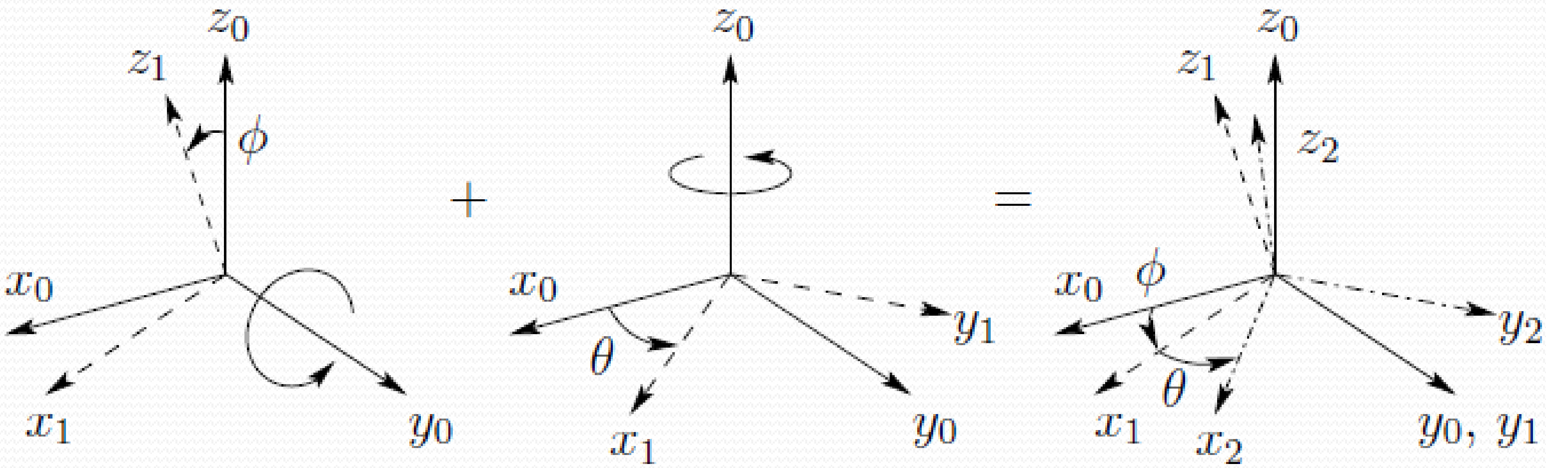
Example composition 2

- Rotation around z axis followed by rotation around the new y axis

$$
\begin{aligned}
R' &= R_{z,\theta} R_{y,\phi} \\
&= \begin{bmatrix} c_\theta & -s_\phi & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}
\end{aligned}
$$

Rotation in Fixed Frame

- Rotation around one axis in a frame followed by another rotation around another axis in the ORIGINAL frame



$$R_2^0 \;=\; R_1^0\left[(R_1^0)^{-1} R R_1^0\right] = R R_1^0$$

Composition of Rotations in Fixed Frame

- Same as in current frame but with reversed order

$$R_2^0 \;=\; R_1^0\left[(R_1^0)^{-1}RR_1^0\right] = RR_1^0$$

- *Why R was not called R_2^1????????*

Example of Rotation composition

Suppose R is defined by the following sequence of basic rotations in the order specified:

1. *A rotation of θ about the current x-axis*
2. *A rotation of ϕ about the current z-axis*
3. *A rotation of α about the fixed z-axis*
4. *A rotation of β about the current y-axis*
5. *A rotation of δ about the fixed x-axis*

In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ and pre- or post-multiply as the case may be to obtain

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \tag{2.24}$$