MTR08114 Robotics Kinematics

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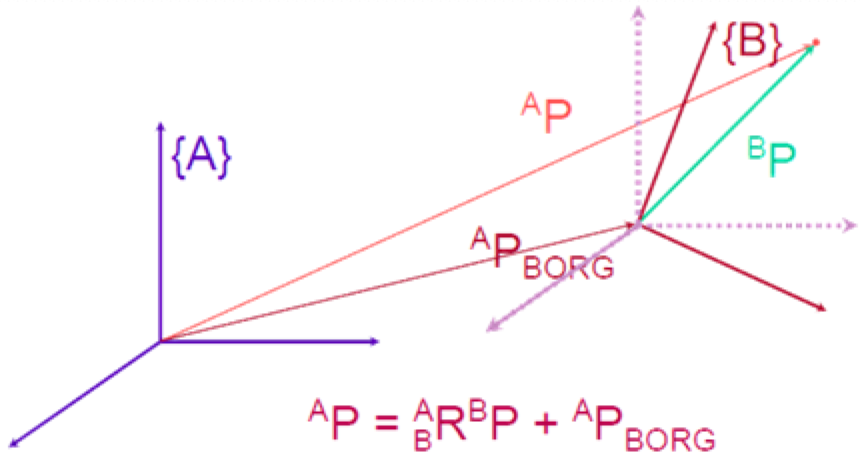
REMINER 1: Rotation Representation

1. Direction Cosines
2. Eular Angle
3. Roll-Pitch-Yaw
4. Axis/Angle

- *Any 3D object possesses only 3 rotational degrees of freedom*

REMINDER 2: Rigid Transformation 2

$$p^0 = R_1^0 p^1 + d_1^0$$



$$\begin{aligned} p^0 &= R_1^0 p^1 + d_1^0 \\ p^1 &= R_2^1 p^2 + d_2^1 \\ p^0 &= R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \\ p^0 &= R_2^0 p^2 + d_2^0 \end{aligned}$$

$$\begin{aligned} R_2^0 &= R_1^0 R_2^1 \\ d_2^0 &= d_1^0 + R_1^0 d_2^1 \end{aligned}$$

REMINDER 3: Inverse Homogeneous

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}$$

$$P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$P^0 = H_1^0 P^1$$

REMINDER 4: Composition Rules

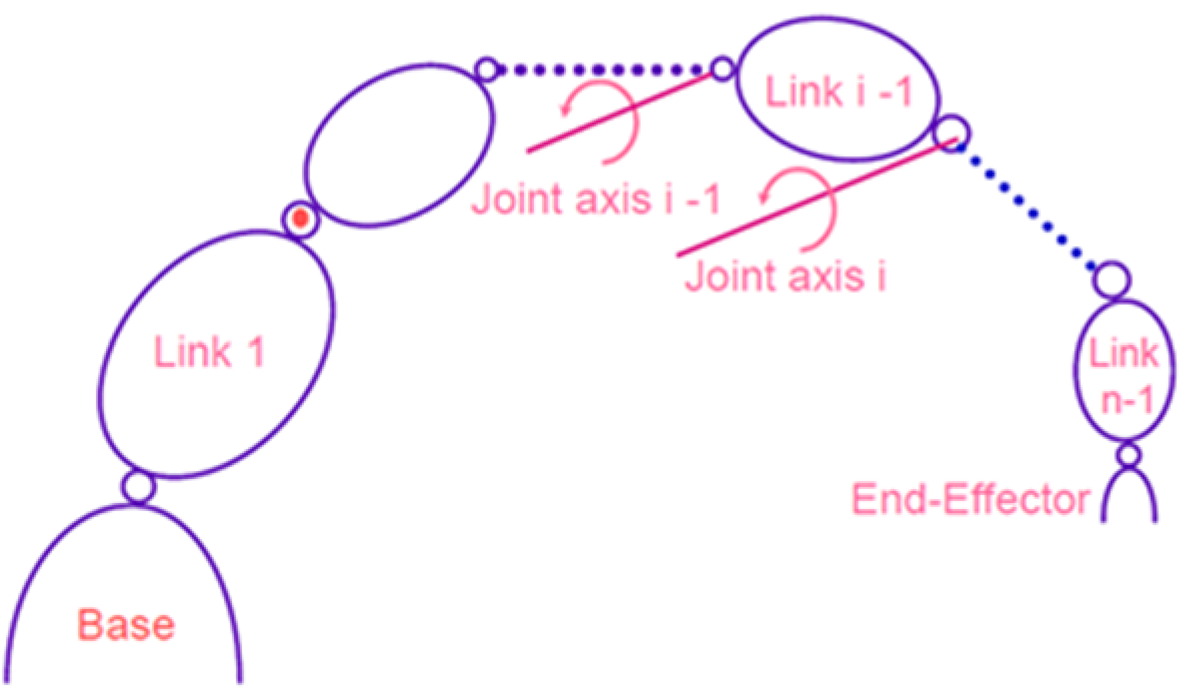
- Around current axis

$$H_2^0 = H_1^0 H$$

- Around fixed axis

$$H_2^0 = H H_1^0$$

Manipulator/ Kinematic Chain



Joint Variables

$$q_i = \begin{cases} d_i & prismatic \\ \theta_i & revolute \end{cases}$$

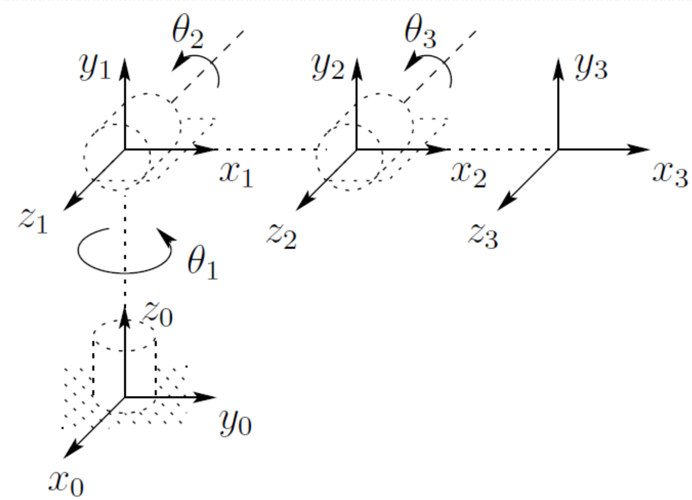
Steps of Kinematic Analysis

1. Attach frame *i* <$o_i x_i y_i z_i$> to link *i*.
   - Coordinates of points in link *i* **in frame *i*** are constant
2. Find the transformation from each frame to the next
   - Origin of frame *i* in frame *i-1*
   - $A_j = T_j^{j-1} = A_j\left(q_j\right)$
3. Find the end effector origin in the base frame
   - $T_n^0 = T_1^0 T_2^1 \ldots\ldots T_{n-1}^{n-2} T_n^{n-1}$

$$T_j^i = A_{i+1} \cdots A_j = \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix}$$

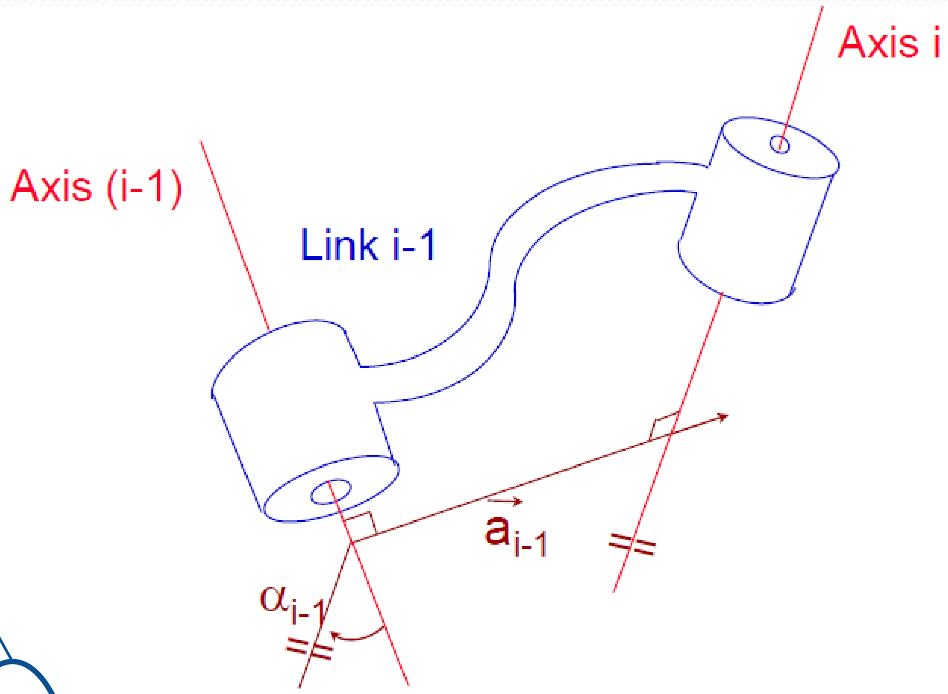
$$R_j^i = R_{i+1}^i \cdots R_j^{j-1}$$

$$o_j^i = o_{j-1}^i + R_{j-1}^i o_j^{j-1}$$



Link Description

Constants once design is done

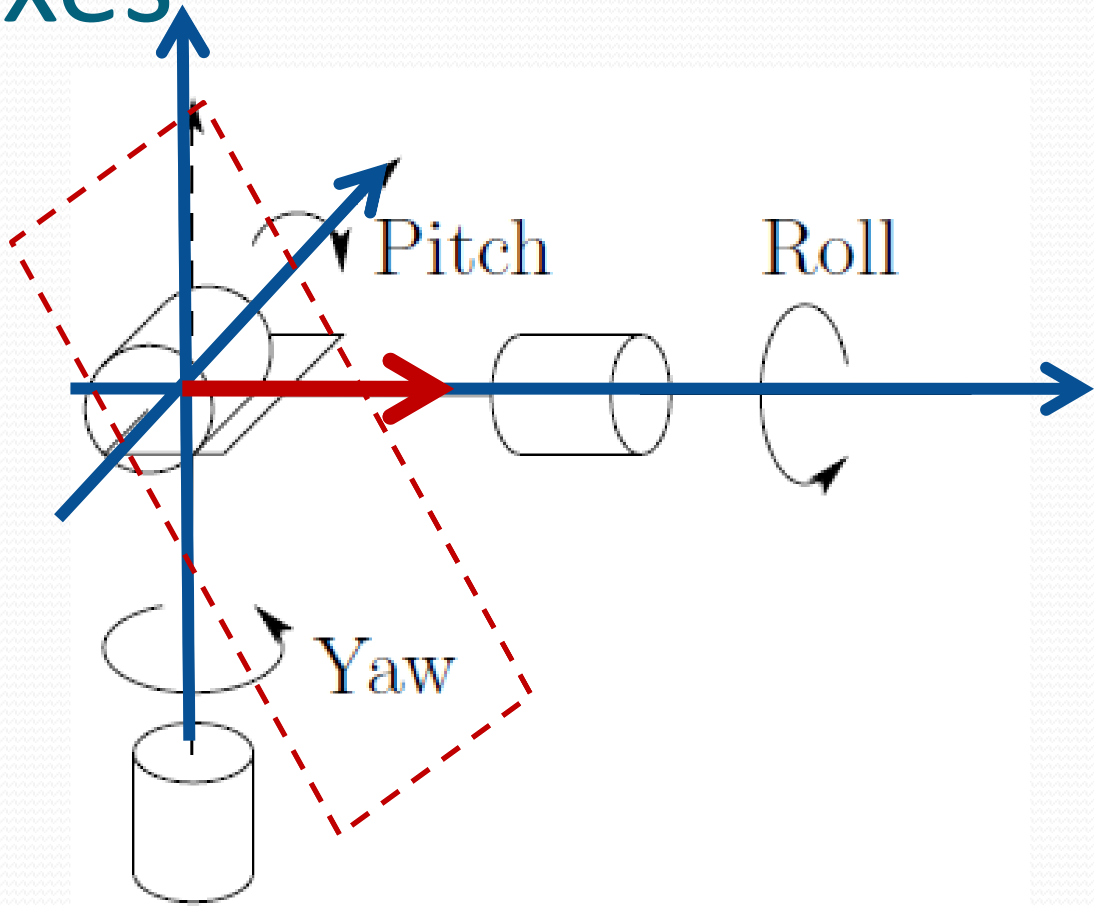


a_{i-1}: Link Length - mutual perpendicular
unique except for parallel axis

α_{i-1}: Link Twist - measured in the right-hand sense about $\vec{a}_{i-1}$

Intersecting axes

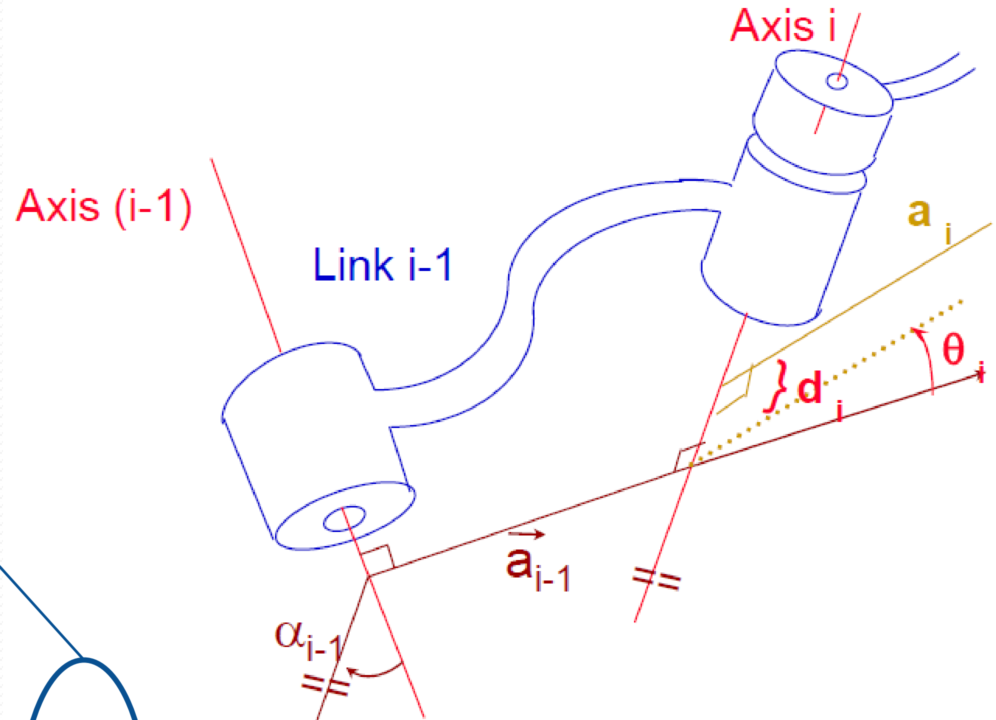
- What is the common normal??????
  - Normal to the plane containing both axes
- Which direction
  - Direction of end effector
- What is the twist
  - Angle in this plane



Joint parameters

VARIABLE once design is done
Constant per configuration

q_i



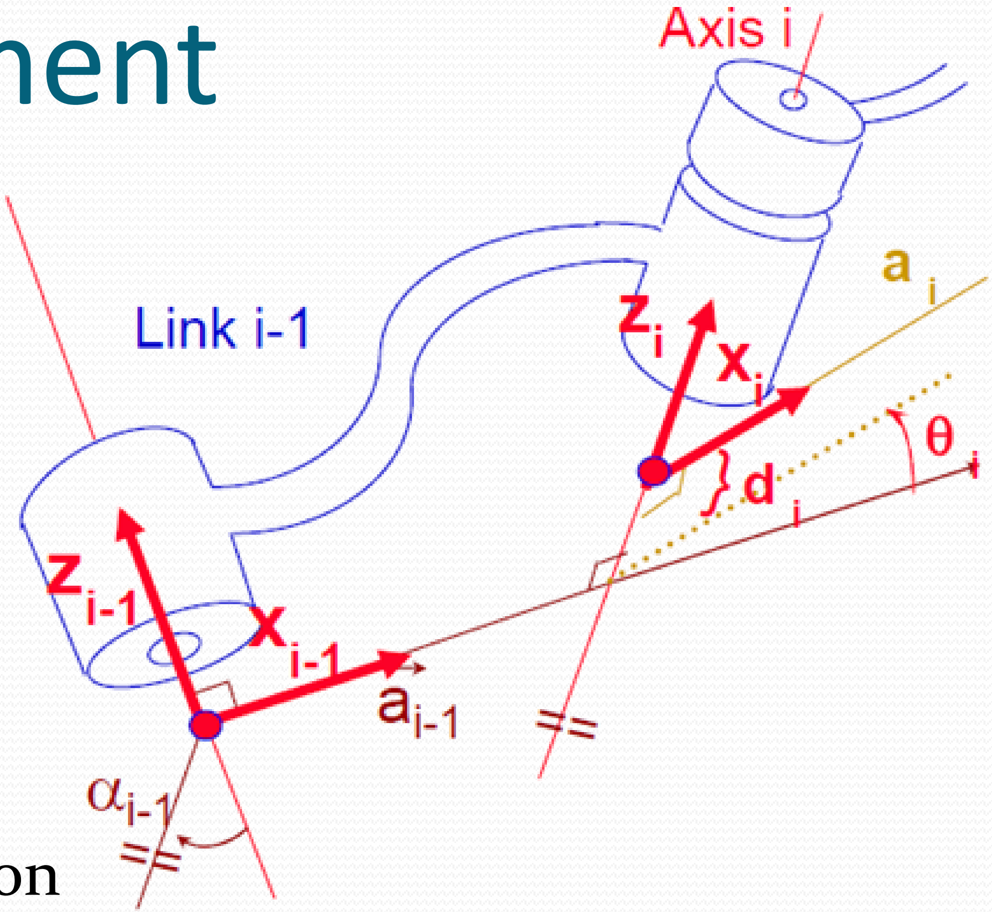
Variable for prismatic ← d_i: Link Offset -- variable if joint i is *prismatic*

Variable for revolute ← θ_i: Joint Angle -- variable if joint i is *revolute*

Denavit-Hartenberg Parameters

- Constants by design
  - Link twist α_i
  - Link length a_i
- Joint parameters
  - Link Offset d_i (variable in prismatic)
  - Joint Angle θ_i (variable in revolute)
- *All frame transformations are functions in these four parameters*

Frame Placement

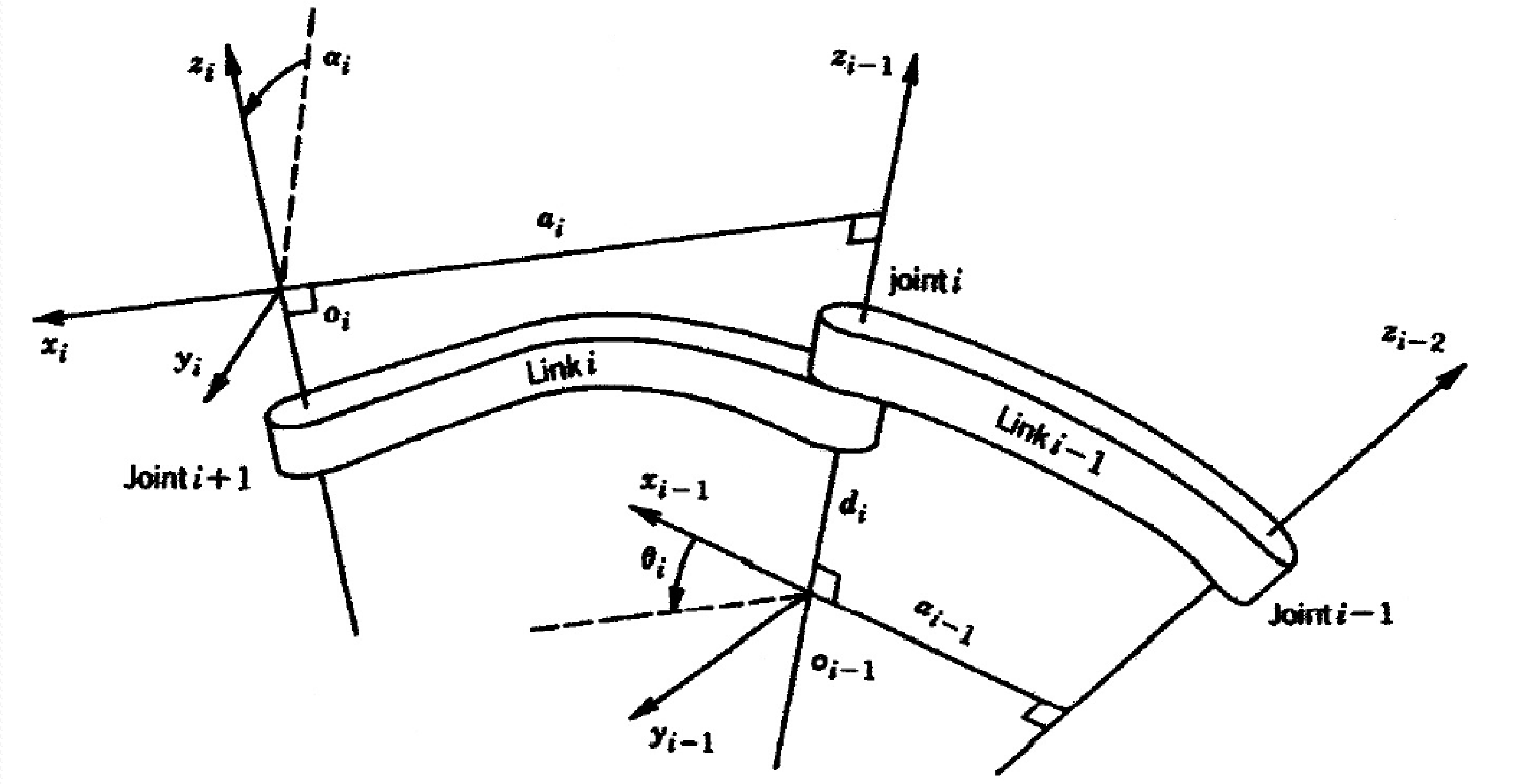


- Z along the axis
- X points to next frame
- Origin in X,Z intersection
- Y using right hand rule

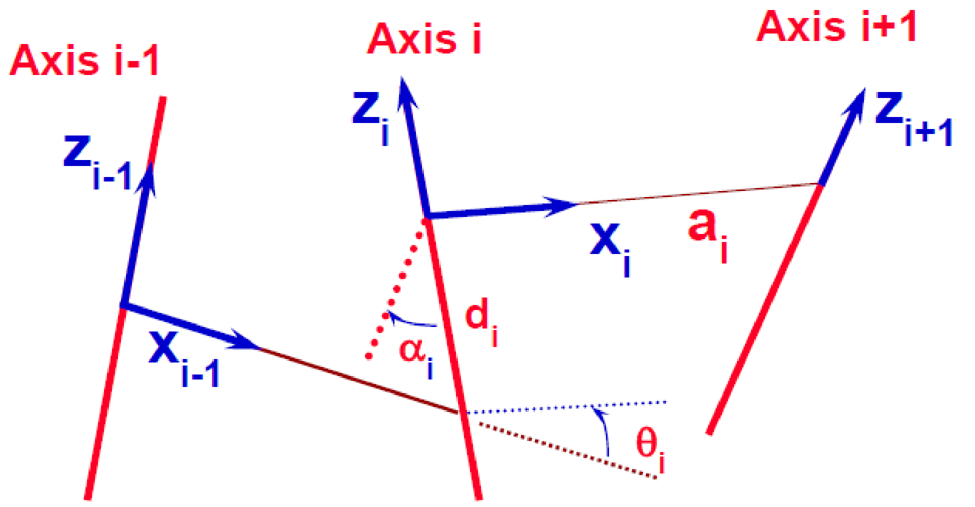
The four six dilemma

- Homogeneous transformation needs 6 parameters
- DH parameters are 4
- Yet DH parameters are enough!!!!
- HOW?
- We have two assumptions:
  - X_i is perpendicular to Z_{i-1}
  - X_i intersects Z_{i-1}

DH (All together)



DH parameters summary



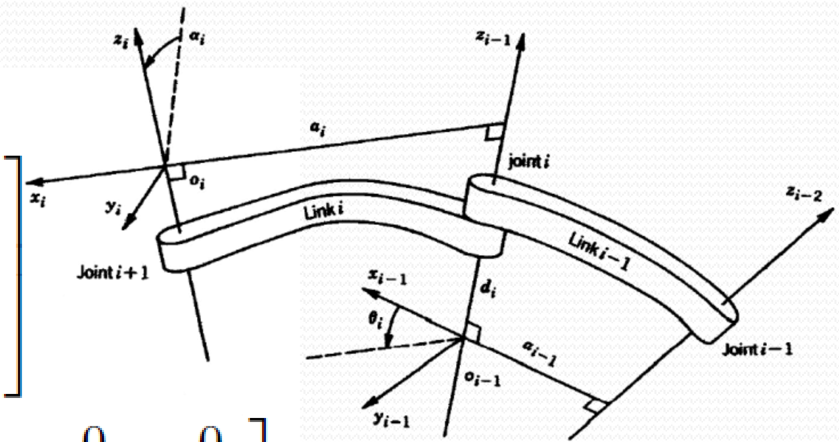
a_i : distance (z_i , z_{i+1}) along x_i

α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

Frame transformation from DH



$$
\begin{aligned}
A_i &= Rot_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}Rot_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Notes about placement

- Z_i and Z_{i-1} are not coplanar
  - Unique common perpendicular (unique a_{i-1} and α_{i-1}).
- Z_i and Z_{i-1} are parallel
  - Infinite number of possible perpendicular. We put the origin as we like to simplify the equations ($\alpha_i = 0$).
- Z_i and Z_{i-1} are intersecting
  - X_i is chosen normal to the common plane in the direction of end effector ($a_i = 0$)

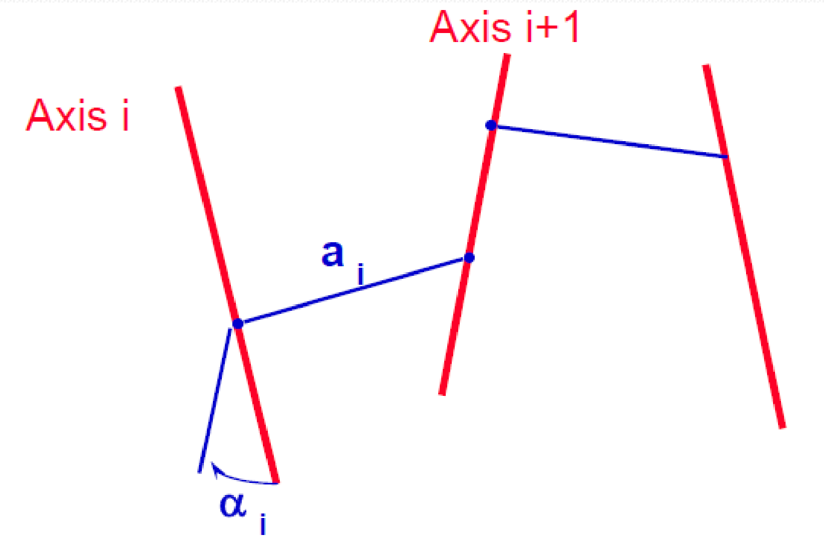
First and Last Frames

- Frames 1 to n correspond to the n joints
- Frame 0 corresponds to the base (no need to be on the base!!)
- Frame n+1 corresponds to end effector (no need to be on it!!)

- *Rule: maximize zeros to simplify forward kinematics*

- How?
  - Put frame 0's origin, X, and Z in the same location as frame 1 when its variable is zero
  - Put frame n+1's origin, X, and Z in the same location as frame n when its variable is zero

First and Last Link's a & α



a_i and α_i depend on joint axes i and i+1

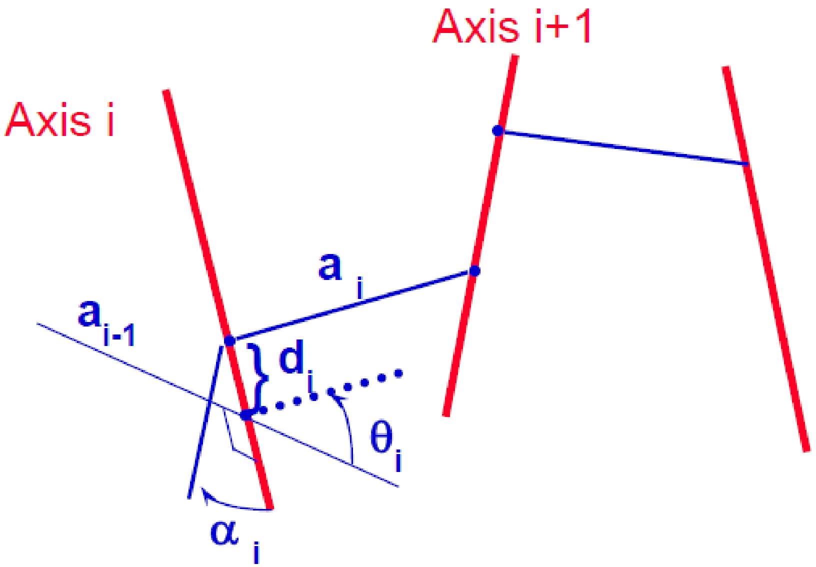
Axes 1 to n: determined

➡ a_1 , a_2 ... a_{n-1} and α_1, α_2 ... α_{n-1}

Convention: $a_0 = a_n = 0$ and $\alpha_0 = \alpha_n = 0$

Used with modification from Osama El Khatib's Standford's Intro to Robotics course material

First & Last Link's d and Θ



θ_i and d_i depend on links i-1 and i

➡ $\theta_2, \theta_3, \ldots, \theta_{n-1}$ and $d_2, d_3, \ldots, d_{n-1}$

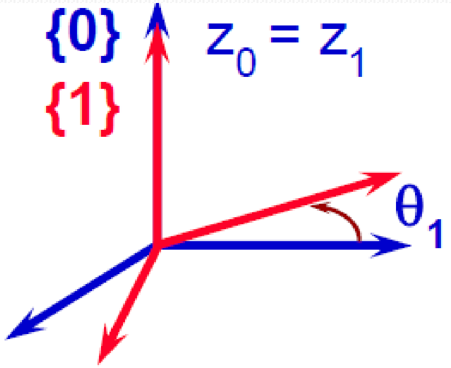
Convention: set the constant parameters to zero

Following joint type: d_1 or $\theta_1 = 0$ and d_n or $\theta_n = 0$

Used with modification from Osama El Khatib's Standford's Intro to Robotics course material

Placement of Base Frame

Revolute



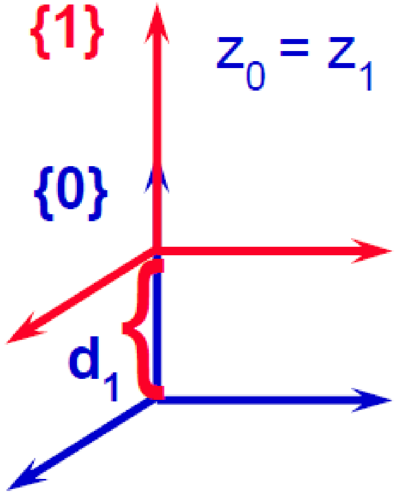
$a_0 = 0$

$\alpha_0 = 0$

$d_1 = 0$

$\theta_1 = 0 \longrightarrow \{0\} \equiv \{1\}$

Prismatic



$a_0 = 0$

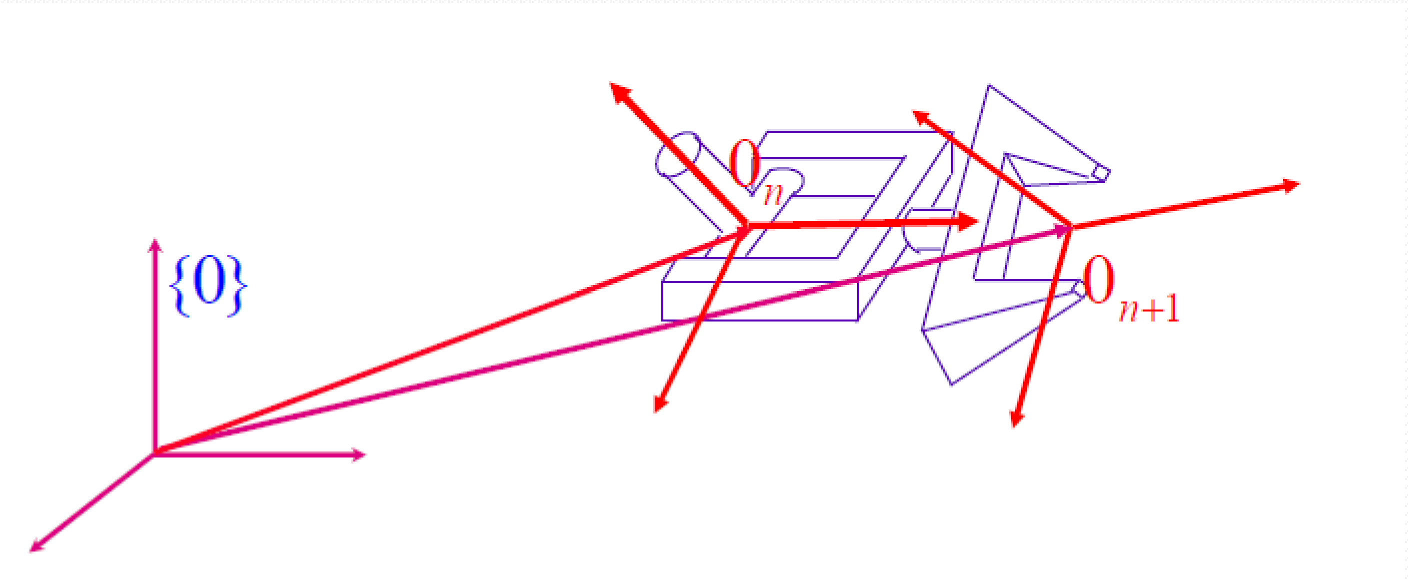
$\alpha_0 = 0$

$\theta_1 = 0$

$d_1 = 0 \longrightarrow \{0\} \equiv \{1\}$

Used with modification from Osama El Khatib's Standford's Intro to Robotics course material

End Effector's frame



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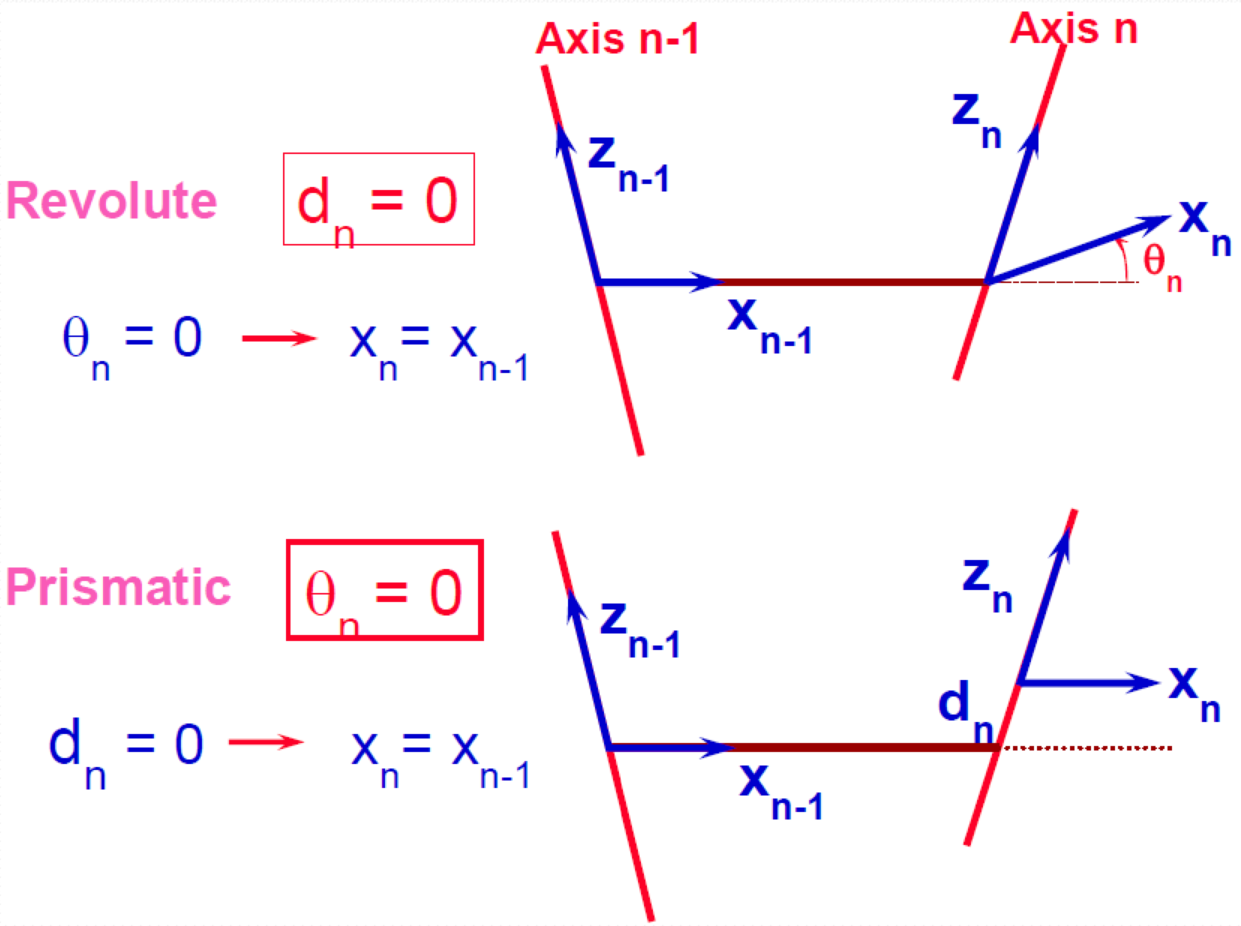
Placement of End Effector

- If specified use the specified frame
- If not specified:
  - Put the origin with the origin of frame n
  - Align Z and X with frame n's Z and X when joint variable n is zero

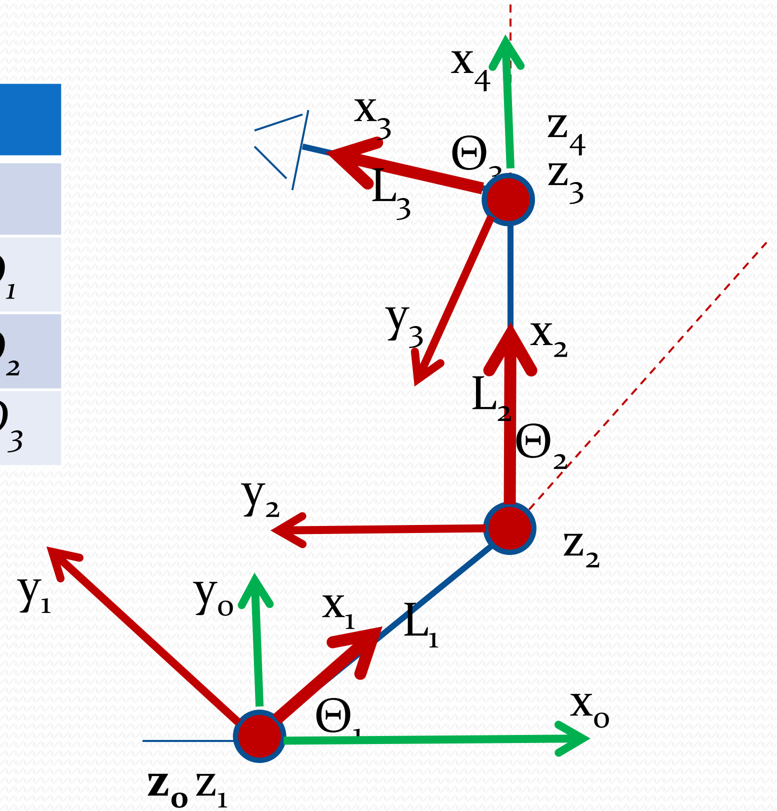
Placement of Last Frame Summary



Example RRR

Link	a_i	α_i	d_i	Θ_i
0	0	0	-	-
1	L_1	0	0	Θ_1
2	L_2	0	0	Θ_2
3	0	0	0	Θ_3

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



1. Place Zs
2. Place Xs
3. Place Origins

a_i : distance (z_i , z_{i+1}) along x_i

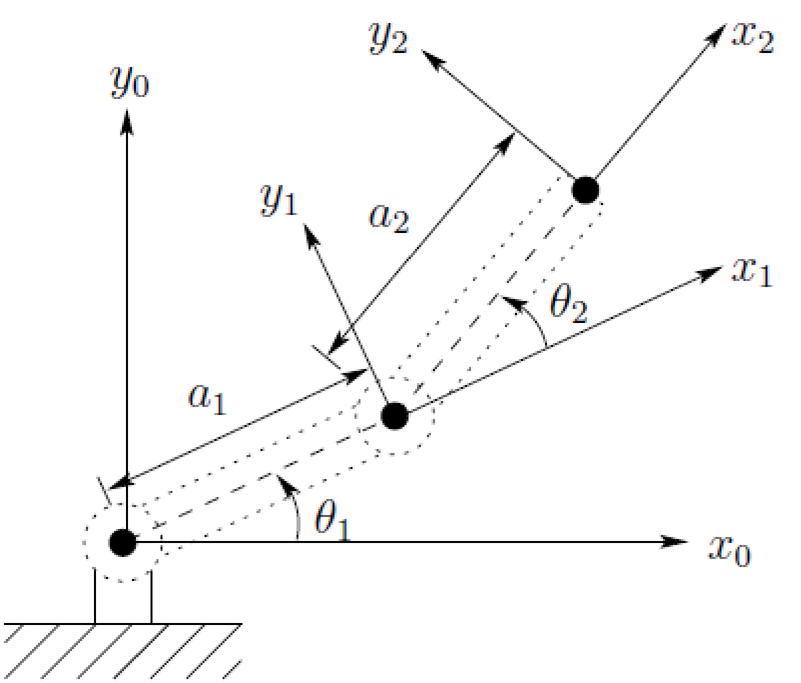
α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

Planar Elbow

Link	a_i	α_i	d_i	Θ_i
0	0	0	-	-
1	a_1	0	0	Θ_1
2	a_2	0	0	Θ_2



$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_ic_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_is_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^0 = A_1$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i : distance (z_i , z_{i+1}) along x_i

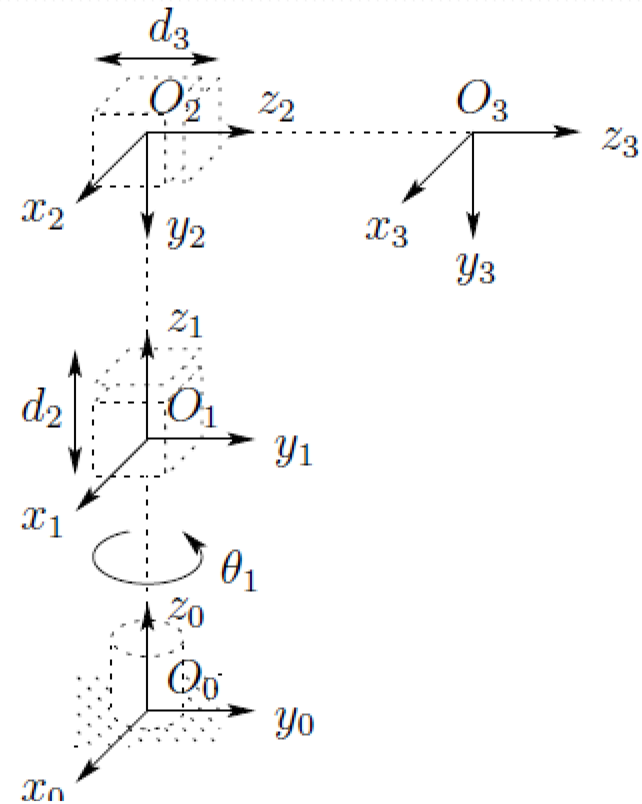
α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

3-Link Cylindrical M.

Link	a_i	α_i	d_i	Θ_i
1	0	0	a_1	Θ_1
2	0	-90	d_2	0
3	0	0	d_3	0



$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i : distance (z_i , z_{i+1}) along x_i

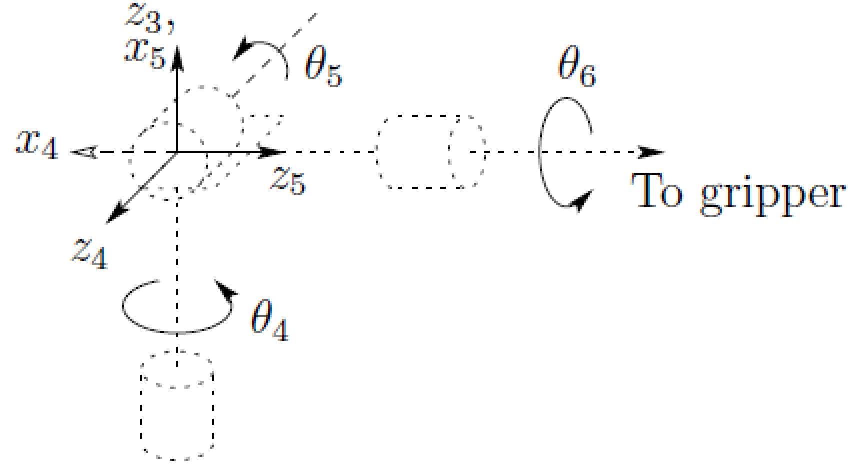
α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

Spherical Wrist

Link	a_i	α_i	d_i	Θ_i
4	0	-90	0	Θ_4
5	0	90	0	Θ_5
6	0	0	a_6	Θ_6



$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} T_6^3 &= A_4 A_5 A_6 \\ &= \begin{bmatrix} R_6^3 & o_6^3 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

a_i : distance (z_i , z_{i+1}) along x_i

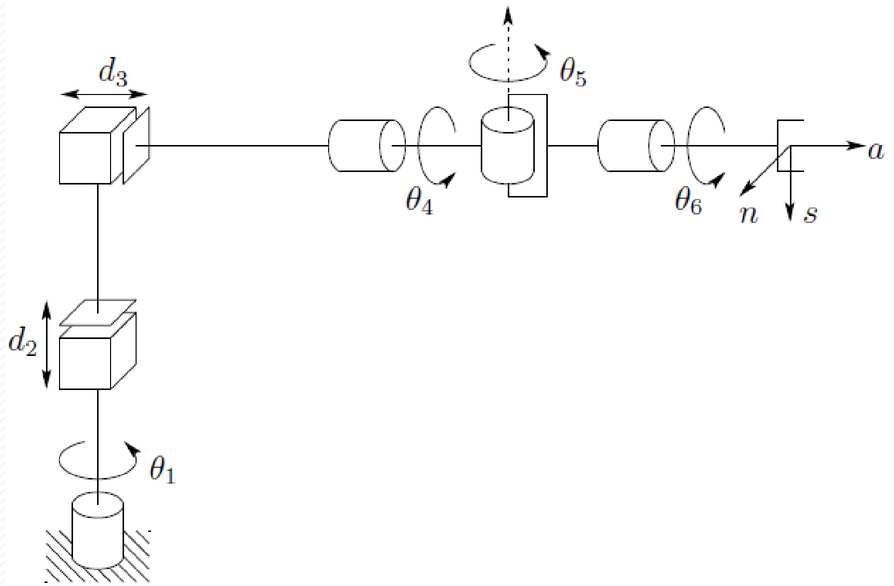
α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

Cylindrical Manipulator with Spherical Wrist

Link	a_i	α_i	d_i	Θ_i
1	0	0	a_1	Θ_1
2	0	-90	d_2	0
3	0	0	d_3	0
4	0	-90	0	Θ_4
5	0	90	0	Θ_5
6	0	0	a_6	Θ_6



$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

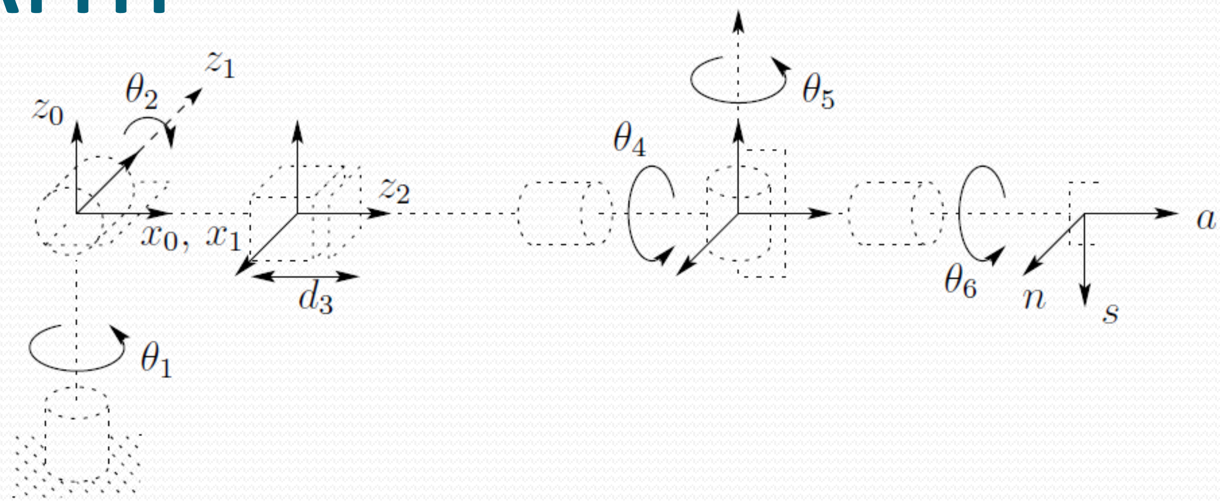
a_i : distance (z_i , z_{i+1}) along x_i

α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

Stanford Arm



Link	a_i	α_i	d_i	Θ_i
1	0	-90	0	Θ_1
2	0	90	a_2	Θ_2
3	0	0	d_3	0
4	0	-90	0	Θ_4
5	0	90	0	Θ_5
6	0	0	a_6	Θ_6

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i : distance (z_i , z_{i+1}) along x_i

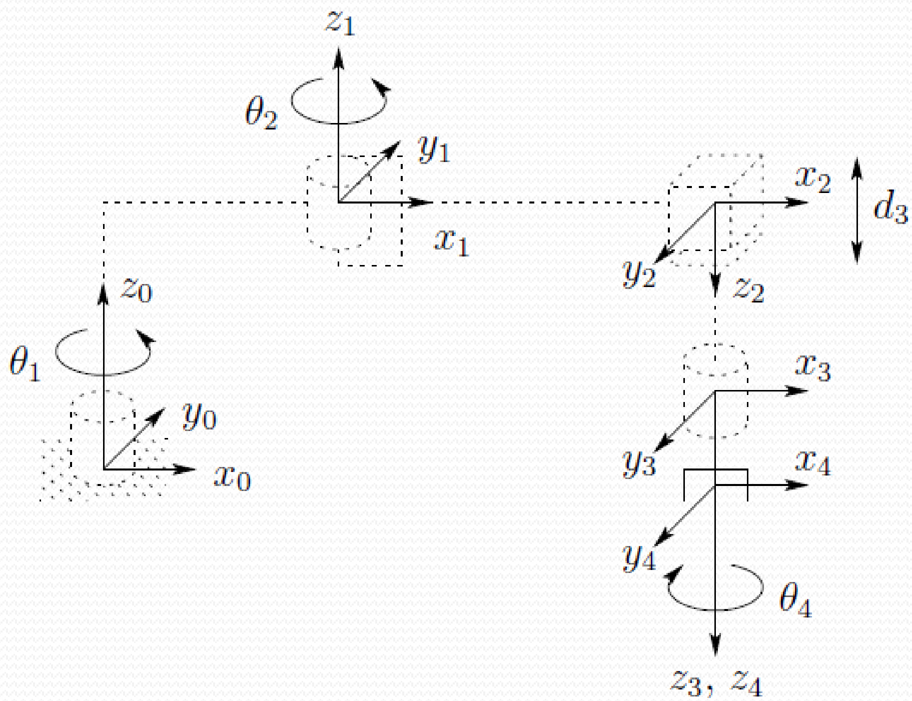
α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

SCARA

Link	a_i	α_i	d_i	Θ_i
1	a_1	0	0	Θ_1
2	a_2	180	0	Θ_2
3	0	0	d_3	0
4	0	0	a_4	Θ_4



$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i : distance (z_i, z_{i+1}) along x_i

α_i : angle (z_i, z_{i+1}) about x_i

d_i : distance (x_{i-1}, x_i) along z_i

θ_i : angle (x_{i-1}, x_i) about z_i