MTR08114 Robotics Velocity Kinematics

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REMINDER 1: General Inverse Kinematic Approach (as we use it)

1. Using desired rotation and displacement of end effector:
   - Find the location of the wrist center
2. Using location of wrist center
   - Find first three parameters (geometrical approach)
   - *This is the solution of the inverse position problem*
3. Using the desired rotation and
   - Find last three parameters (Eular Angles)
   - *This is the solution of the inverse orientation problem*

REMINDER 2: Inverse Position (Geometric)

- Given o_c^0
  - Find q_1, q_2, q_3

- Steps to find q_i :
  1. Project the manipulator onto x_{i-1}-y_{i-1} plane
  2. Solve a simple trigonometry problem

- Why geometric approach?
  - Simple
  - Applies to MOST manipulators

REMINDER 3: Inverse Orientation

Using the desired rotation and loc.of wrist center

Find last three parameters

Can be interpreted as finding Eular Angles

$$\theta_4 = \phi$$

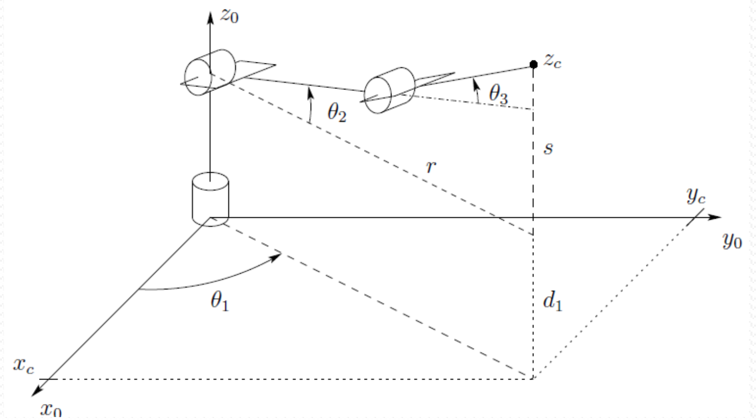
$$\theta_5 = \theta$$

$$\theta_6 = \psi$$

REMINDER 4: Elbow Manipulator – Full Solution

Given

$$o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix}; \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$



$$\begin{aligned} x_c &= o_x - d_6 r_{13} \\ y_c &= o_y - d_6 r_{23} \\ z_c &= o_z - d_6 r_{33} \end{aligned}$$

$$\begin{aligned} \theta_1 &= \operatorname{atan2}(x_c, y_c) \\ \theta_2 &= \operatorname{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \operatorname{atan2}(a_2 + a_3 c_3, a_3 s_3) \\ \theta_3 &= \operatorname{atan2}\left(D, \pm\sqrt{1 - D^2}\right), \\ & \quad \text{where } D = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2 a_2 a_3} \\ \theta_4 &= \operatorname{atan2}(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, \\ & \qquad -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}) \\ \theta_5 &= \operatorname{atan2}\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}\right) \\ \theta_6 &= \operatorname{atan2}(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22}) \end{aligned}$$

Velocity Kinematics

- Relation between end effector's linear and angular velocities and joint velocities.

- This is defined by the Jacobian (one of the most important concepts in robot motion)

- Steps:
  - Understand velocity and its transfer with moving frames!!
  - Derive Jacobian
  - Understand singularities

Angular Velocity: FIXED AXIS

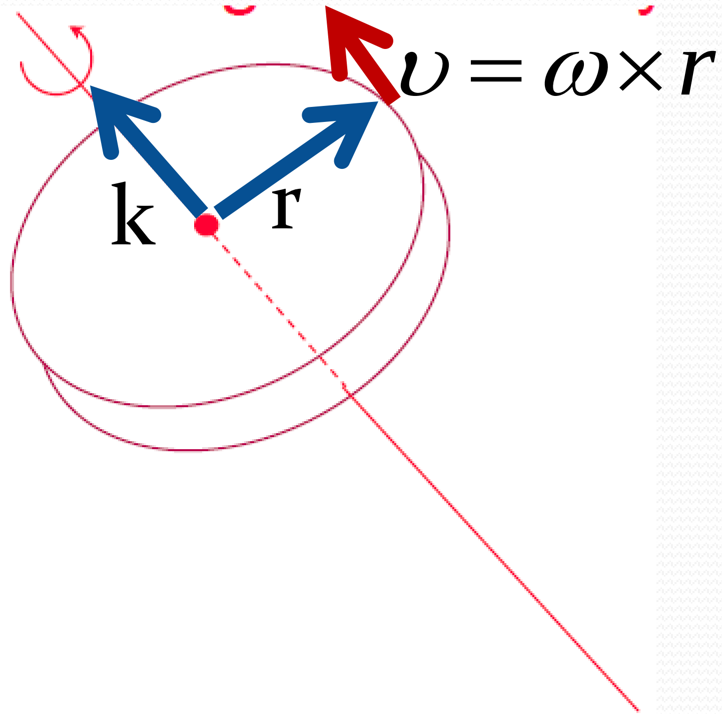
- Angular Velocity (describes a frame)

$$\omega = \dot{\theta} k$$

- Linear velocity (describes a point)

$$\upsilon = \omega \times r$$

- Angular velocity if fixed for the wh body
- Linear velocity depends on the distance between the point and the axis of rotation



- *How to represent angular velocity?*

Skew Matrix

- A square matrix S is said to be a skew matrix iff

$$S^T + S = 0$$

- All 3×3 skew matrices are said to belong to *so(3)*
- From definition: $s_{ij} = -s_{ji}$ $\qquad s_{ii} = 0$

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

- ONLY 3 INDEPENDENT VALUES (Rank 3)

Skew Matrix of a vector

- For a vector

$$a = [a_1, a_2, a_3]$$

- The corresponding Skew Matrix is

$$S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Basic Skew Matrices

$$S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S(i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad S(j) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$S(k) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Properties of Skew Matrices

- Cross product rule:

$$S(a)p = a \times p$$

- Linearity rule:

$$S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$$

- First Rotation Rule

$$R(a \times p) = Ra \times Rp$$

- Second Rotation Rule

$$RS(a)R^T = S(Ra)$$

Relation between S and R

$$RR^T = I$$

$$\therefore \frac{dR}{d\theta}R^T + R\frac{dR^T}{d\theta} = 0$$

$$define \quad \mathrm{S}=\frac{dR}{d\theta}R^T$$

$$\therefore S^T = \left(\frac{dR}{d\theta}R^T\right)^T = R\frac{dR^T}{d\theta}$$

$$\therefore S + S^T = 0$$

where

$$SR = \frac{dR}{d\theta}$$

Example

$$R = R_{x,\theta}$$

$$S = \frac{dR}{d\theta} R^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(i)$$

Derivative of a Rotation Matrix

$$\frac{dR_{k,\theta}}{d\theta} = S(k)R_{k,\theta}$$

So What?

- We represent orientation by a rotation matrix
- We represent change in orientation (angular velocity) by a Skew matrix
- They are related and the relation is:

$$\frac{dR_{k,\theta}}{d\theta} = S(k)R_{k,\theta}$$

What do we have now?

$$R(\theta) \in SO(3)$$

$$S \in so(3)$$

$$R(\theta)R(\theta)^T = I$$

$$S + S^T = 0$$

$$SR(\theta) = \frac{dR(\theta)}{d\theta}$$

Variable rotation axis →

$$R(t) \in SO(3)$$

$$S(t) \in so(3)$$

$$R(t)R^T(t) = I$$

$$S(t) + S^T(t) = 0$$

$$S(t)R(t) = \dot{R}(t)$$

What is S(t)

$$S(t)R(t) = \dot{R}(t)$$

- For every S there is a vector that S is the cross product operator for it

$$S(\omega(t))R(t) = \dot{R}(t)$$

Angular Velocity

Example

Suppose that $R(t) = R_{x,\theta(t)}$. *Then* $\dot{R}(t)$ *is computed using the chain rule as*

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \dot{\theta}S(i)R(t) = S(\omega(t))R(t) \qquad (4.20)$$

in which $\omega = i\dot{\theta}$ *is the* **angular velocity**. *Note, here* $i = (1, 0, 0)^T$.

- $\omega^i_{j,k}$ represents the angular velocity corresponding to the derivative of the rotation matrix R^j_k expressed in frame *i*.
- $\omega_{j,k}$ represents the angular velocity corresponding to the derivative of the rotation matrix R^j_k expressed in frame *o*.
- ω_k represents the angular velocity corresponding to the derivative of the rotation matrix R^o_k expressed in frame *o*.

Combining Angular Velocities

- Assume that one frame (0) is fixed and frame (1) is rotating then frame (2) is also rotating:

$$R_2^0(t) = R_1^0(t)R_2^1(t)$$

$$\dot{R}_2^0 = S(\omega_{0,2}^0)R_2^0$$

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

$$\dot{R}_1^0 R_2^1 = S(\omega_{0,1}^0)R_1^0 R_2^1 = S(\omega_{0,1}^0)R_2^0$$

$$\begin{aligned} R_1^0 \dot{R}_2^1 &= R_1^0 S(\omega_{1,2}^1)R_2^1 \\ &= R_1^0 S(\omega_{1,2}^1){R_1^0}^T R_1^0 R_2^1 = S(R_1^0 \omega_{1,2}^1)R_1^0 R_2^1 \\ &= S(R_1^0 \omega_{1,2}^1)R_2^0. \end{aligned}$$

$$S(\omega_2^0)R_2^0 = \{S(\omega_{0,1}^0) + S(R_1^0 \omega_{1,2}^1)\}R_2^0$$

$$S(a) + S(b) = S(a+b)$$

$$\omega_2^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1$$

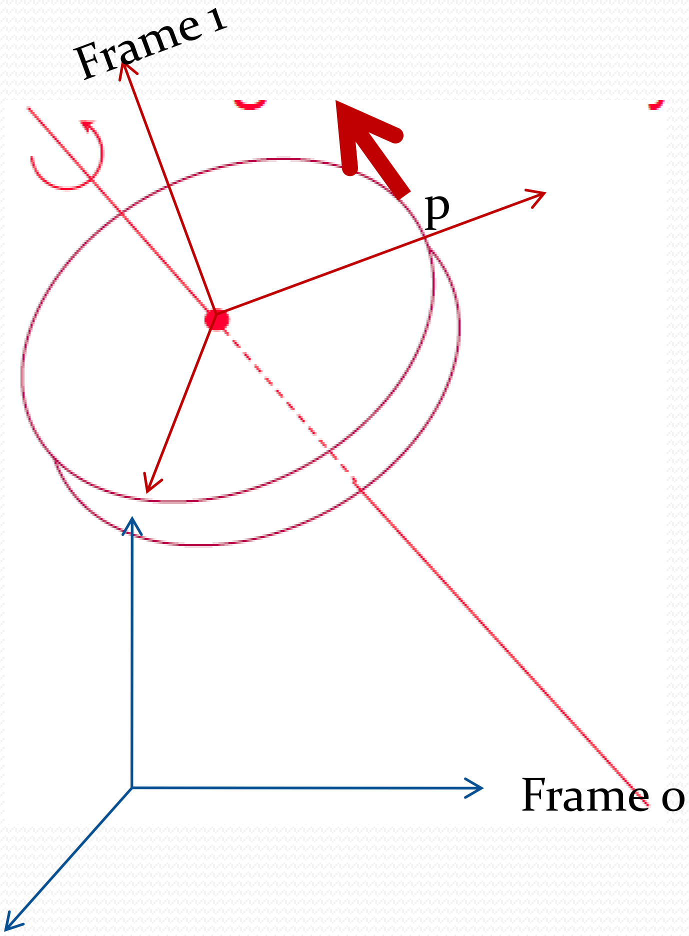
Linear Velocity of a Point in a rotating frame

$$\upsilon = \omega \times p$$

- Rotation Only:

$$p^0 = R_1^0(t)p^1.$$

$$\begin{aligned}\dot{p}^0 &= \dot{R}_1^0(t)p^1 + R_1^0(t)\dot{p}^1 \\ &= S(\omega^0)R_1^0(t)p^1 \\ &= S(\omega^0)p^0 = \omega^0 \times p^0\end{aligned}$$



Linear Velocity of a Point in a translating frame

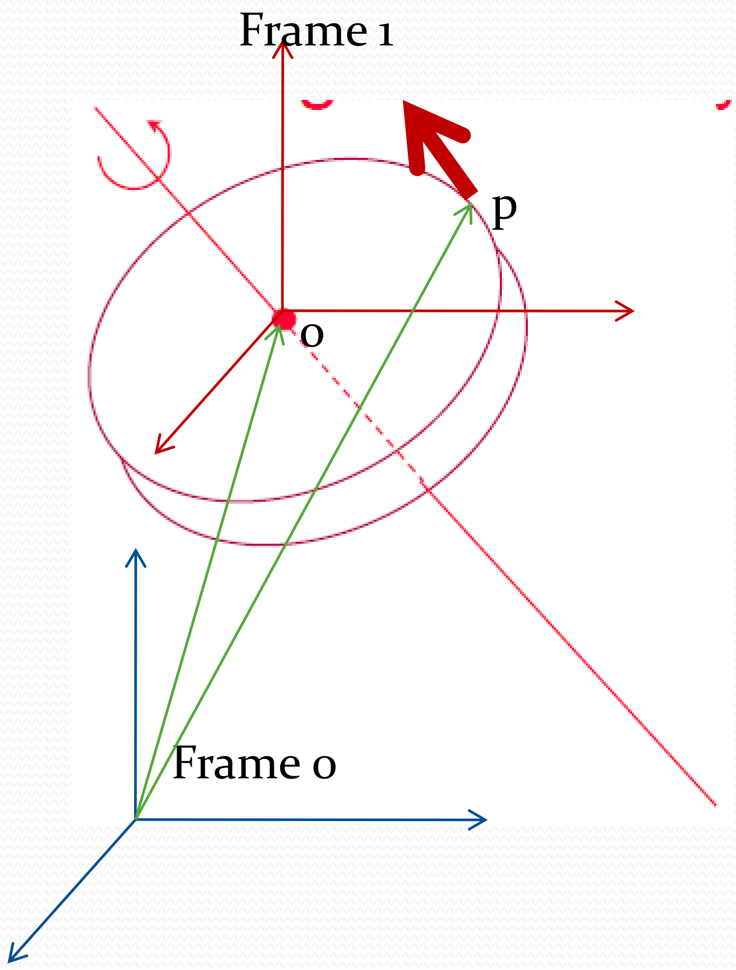
$$\upsilon = \omega \times p$$

- Translating Only:

$$p^0 = p^1 + o$$

$$\dot{p}^0 = \dot{p}^1 + \dot{o}$$

$$\dot{p}^0 = \upsilon$$



Linear Velocity of a Point in a moving frame

$$\upsilon = \omega \times p$$

- Translating And Rotating:

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & o_1^0(t) \\ 0 & 1 \end{bmatrix}$$

$$p^0 = Rp^1 + o$$

$$\begin{aligned} \dot{p}^0 &= \dot{R}p^1 + \dot{o} \\ &= S(\omega)Rp^1 + \dot{o} \\ &= \omega \times r + v \end{aligned}$$