MTR08114 Robotics Dynamics

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REMINDER 1: The whole Jacobian (METHOD 1)

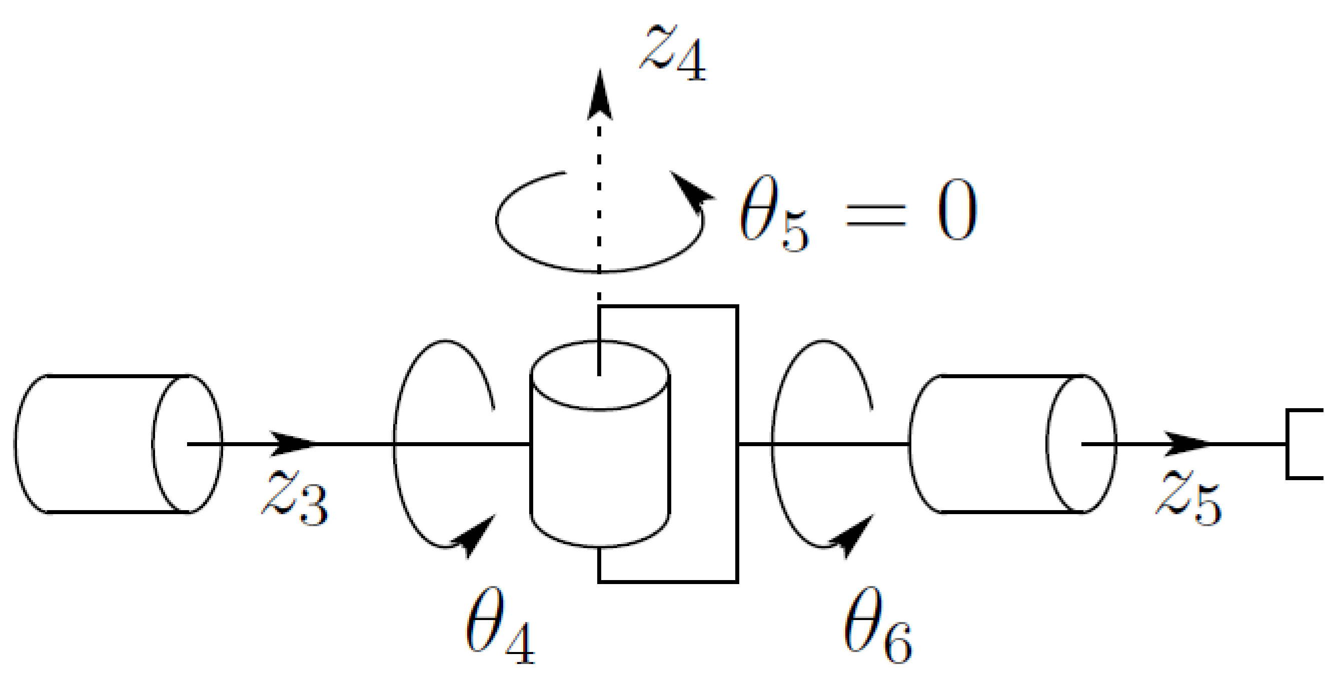
- Revolute Joint

$$J_i = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}$$

- Prismatic Joint

$$J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}$$

REMINDER 2: Wrist Singularities



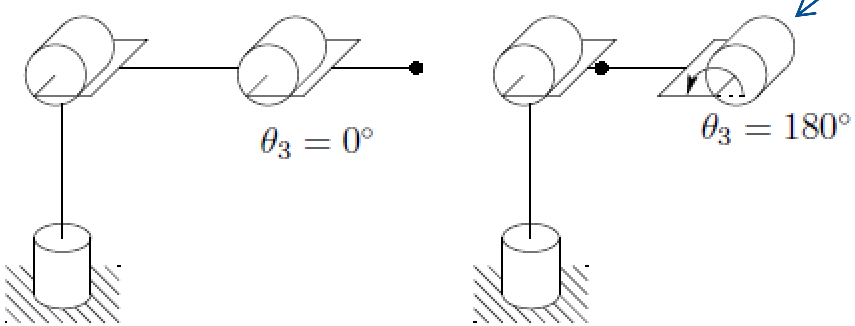
- Whenever z_3 and z_5 are aligned

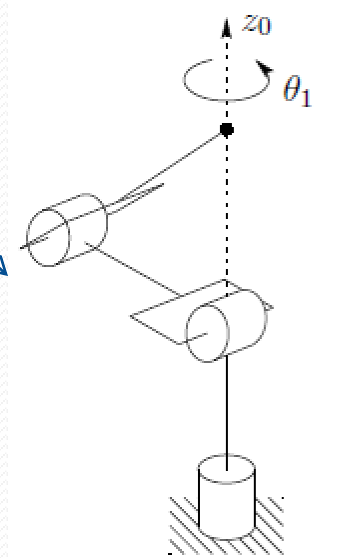
- Prove it

REMINDER 3: Elbow Singularity

$$J_{11} = \begin{bmatrix} -a_2 s_1 c_2 - a_3 s_1 c_{23} & -a_2 s_2 c_1 - a_3 s_{23} c_1 & -a_3 c_1 s_{23} \\ a_2 c_1 c_2 + a_3 c_1 c_{23} & -a_2 s_1 s_2 - a_3 s_1 s_{23} & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{bmatrix}$$

$$\det J_{11} = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23}).$$





REMINDER 4: Resolved Motion Rate Control (Whitney 1972)

$$\delta x = J(\theta)\delta\theta$$

Outside singularities

$$\delta\theta = J^{-1}(\theta)\delta x$$

Arm at Configuration θ

$$x = f(\theta)$$

$$\delta x = x_d - x$$

$$\delta\theta = J^{-1}\delta x$$

$$\boxed{\theta^{+} = \theta + \delta\theta}$$

From Osama Khatib

REMINDER 5: How to calculate J^+

- Most difficult method (from definition):

$$J^+ = \left(J^T J\right)^{-1} J^T$$

- Simplest Method (SVD):

$$J = U \sum V^T \qquad \sigma_{ij}^+ = 1/\sigma_{ij}, \quad \text{when } \sigma_{ij} \neq 0$$

$$J^+ = U^T \sum^+ V$$

REMINDER 6: Manipulability

$$\mu = \prod_{i=1}^{m} \sigma_{ii}$$

If the robot is not redundant (n<=6)

$$\mu = \left| \det \quad J \right|$$

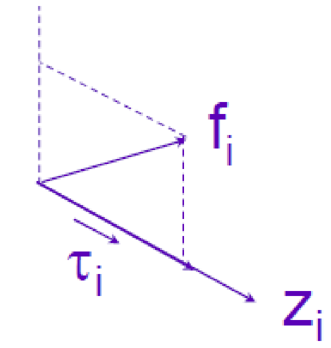
REMINDER 7: Velocity Force Duality

$$\zeta = J\dot{\theta}$$

$$\tau = J^T F$$

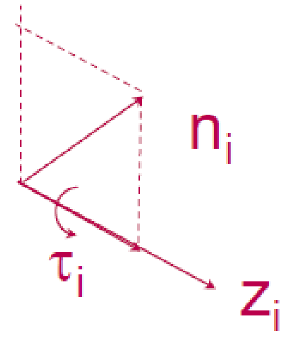
$$F = \begin{bmatrix} f \\ n \end{bmatrix}$$

REMINDER 8: How to do the elimination



Prismatic Joint

$$\tau_i = f_i^T Z_i$$



Revolute Joint

$$\tau_i = n_i^T Z_i$$

Algorithm

$${}^n f_n = {}^n f$$

$${}^n n_n = {}^n n + {}^n P_{n+1} \times {}^n f$$

$${}^i f_i = {}_{i+1}^{\,i}R \,.\, {}^{i+1} f_{i+1}$$

$${}^i n_i = {}_{i+1}^{\,i}R \,.\, {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i$$

From Osama Khatib

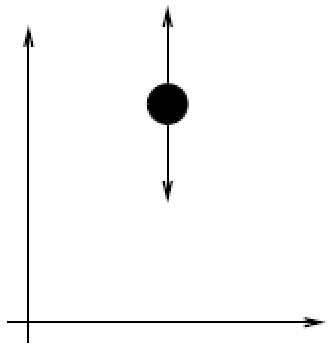
What is dynamics

- Relationship between forces and motion

- Two approaches:
  - Traditional Newton laws approach
  - Eular-Lagrange formulation

- In robotics we use Eular-Lagrange formulation because it is MUCH easier for complex objects.

Motivation (1D system)



$$m\ddot{y} = f - mg$$

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{y}}$$

$$mg = \frac{\partial}{\partial y}(mgy) = \frac{\partial \mathcal{P}}{\partial y}$$

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2}m\dot{y}^2 - mgy$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{K}}{\partial \dot{y}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial \mathcal{P}}{\partial y}$$

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f}$$

Eular- Lagrange Equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \quad i = 1, \ldots, n$$

Kinetic Energy

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{I} \omega$$

Translation

Rotation

Inertia Tensor

Calculating Inertia Tensor in Fixed Frame

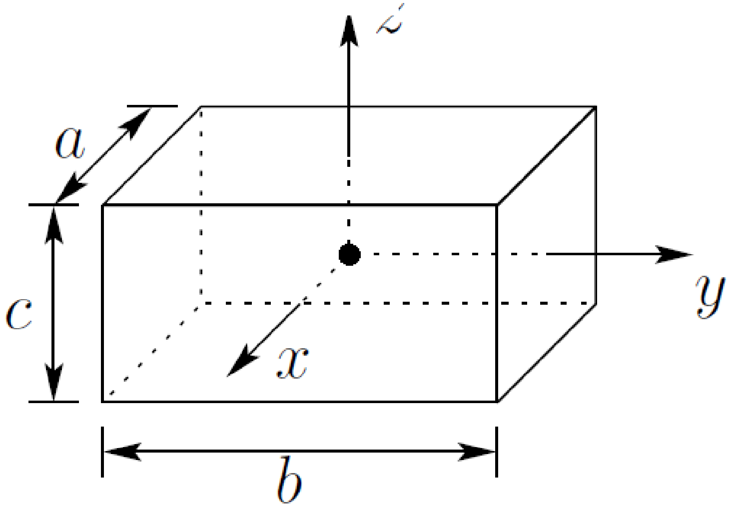
$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$\begin{aligned}
I_{xx} &= \int\int\int (y^2 + z^2)\rho(x, y, z)dx\ dy\ dz \\
I_{yy} &= \int\int\int (x^2 + z^2)\rho(x, y, z)dx\ dy\ dz \\
I_{zz} &= \int\int\int (x^2 + y^2)\rho(x, y, z)dx\ dy\ dz \\
I_{xy} = I_{yx} &= -\int\int\int xy\rho(x, y, z)dx\ dy\ dz \\
I_{xz} = I_{zx} &= -\int\int\int xz\rho(x, y, z)dx\ dy\ dz \\
I_{yz} = I_{zy} &= -\int\int\int yz\rho(x, y, z)dx\ dy\ dz
\end{aligned}$$

Example (Uniform Solid)

$$I_{xx} = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2+z^2)\rho(x,y,z)dx\ dy\ dz = \rho\frac{abc}{12}(b^2+c^2)$$

$$I_{yy} = \rho\frac{abc}{12}(a^2+c^2) \quad ; \quad I_{zz} = \rho\frac{abc}{12}(a^2+b^2)$$



Moving Inertia Tensor Between Frames

$$\mathcal{I} = RIR^T$$

Kinetic Energy for N Links Robot

$$v_i = J_{v_i}(q)\dot{q}, \qquad \omega_i = J_{\omega_i}(q)\dot{q}$$

$$K = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \left[m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q) \right] \dot{q}$$

$$\boxed{K = \frac{1}{2}\dot{q}^T D(q)\dot{q}}$$

Potential Energy for N Link Robot

$$P_i = g^T r_{ci} m_i$$

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} g^T r_{ci} m_i$$

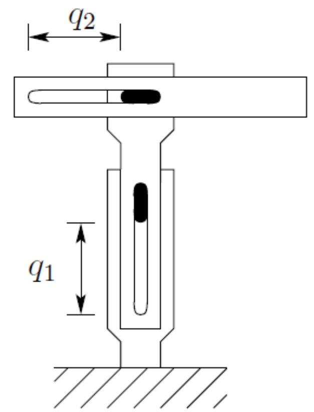
Eular-Lagrange Equation of a Robot

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) \quad = \quad \tau$$

$$\begin{aligned} c_{kj} &= \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i \\ &= \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_j} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i \end{aligned}$$

Christoffel Symbols of the first order

Example 2D Cartesian (PP)



$$v_{c1} = J_{v_{c1}} \dot{q}$$

$$J_{v_{c1}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$v_{c2} = J_{v_{c2}} \dot{q}$$

$$J_{v_{c2}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$K = \frac{1}{2} \dot{q}^T \left\{ m_1 J_{v_c}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} \right\} \dot{q}$$

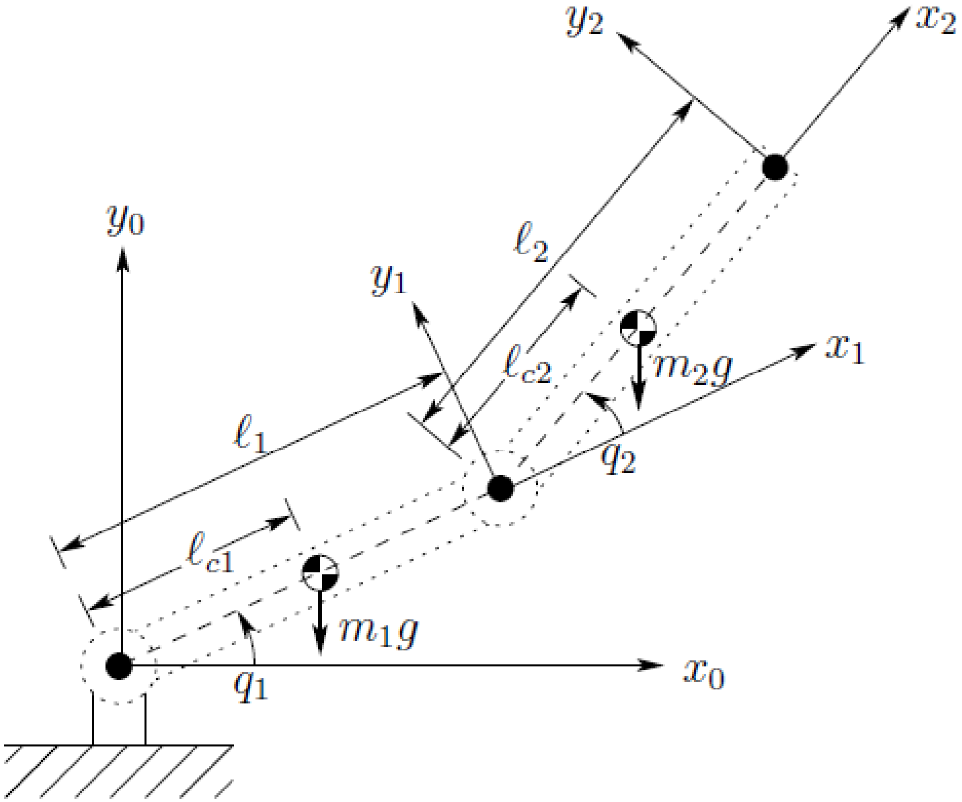
$$D = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$P = g(m_1 + m_2) q_1$$

$$\phi_1 = \frac{\partial P}{\partial q_1} = g(m_1 + m_2), \qquad \phi_2 = \frac{\partial P}{\partial q_2} = 0$$

$$\begin{aligned} (m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) &= f_1 \\ m_2 \ddot{q}_2 &= f_2 \end{aligned}$$

Planar Elbow



$$v_{c1} = J_{v_{c1}} \dot{q}$$

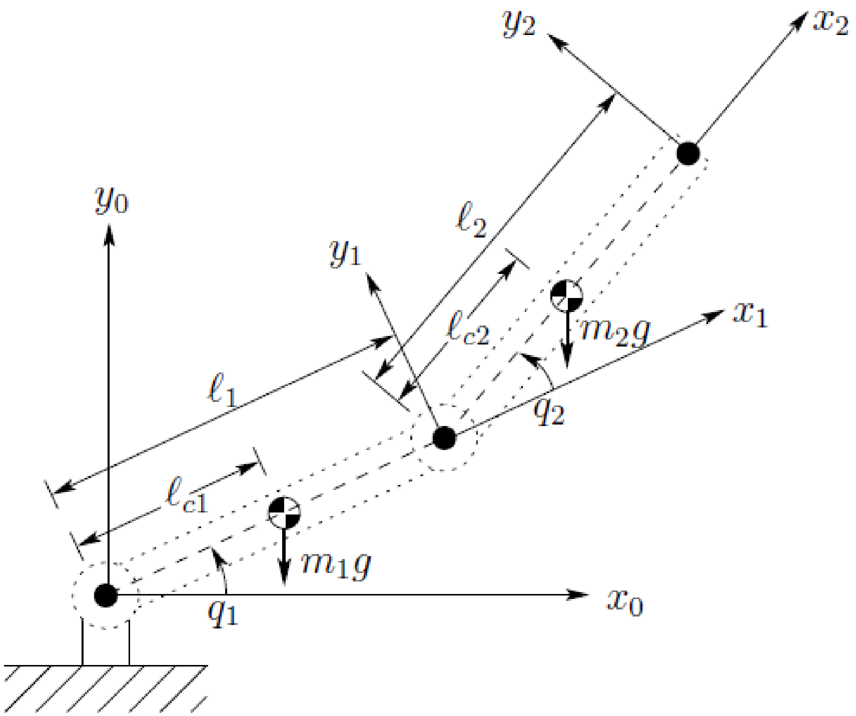
$$J_{v_{c1}} = \begin{bmatrix} -\ell_c \sin q_1 & 0 \\ \ell_{c1} \cos q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$v_{c2} = J_{v_{c2}} \dot{q}$$

$$J_{v_{c2}} = \begin{bmatrix} -\ell_1 \sin q_1 - \ell_{c2} \sin(q_1 + q_2) & -\ell_{c2} \sin(q_1 + q_2) \\ \ell_1 \cos q_1 + \ell_{c2} \cos(q_1 + q_2) & \ell_{c2} \cos(q_1 + q_2) \\ 0 & 0 \end{bmatrix}$$

$$\frac{1}{2} m_1 v_{c1}^T v_{c1} + \frac{1}{2} m_2 v_{c2}^T v_{c2} = \frac{1}{2} \dot{q} \left\{ m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} \right\} \dot{q}$$

Planar Elbow



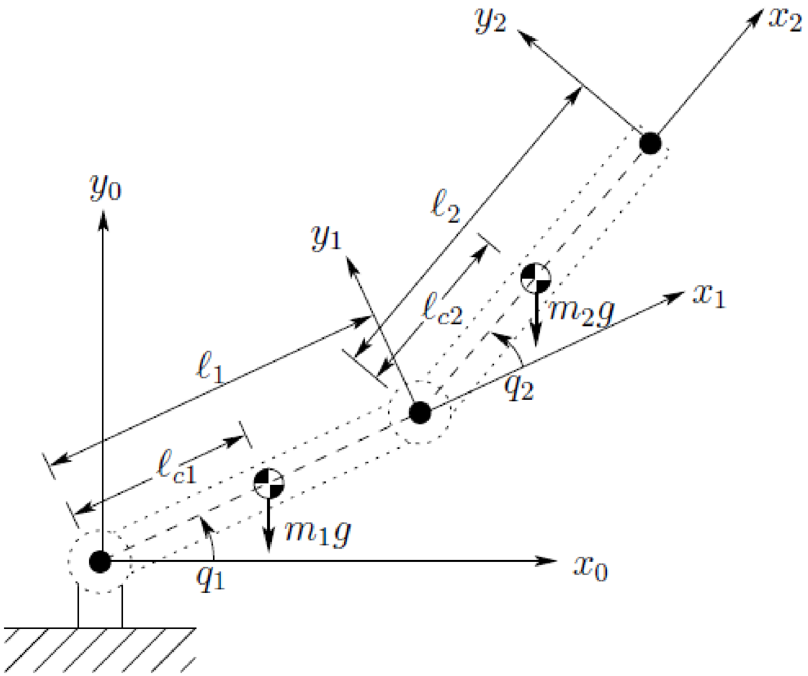
$$\omega_1 = \dot{q}_1 k, \qquad \omega_2 = (\dot{q}_1 + \dot{q}_2)k$$

$$\frac{1}{2}\dot{q}^T \left\{ I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \dot{q}$$

$$D(q) = m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

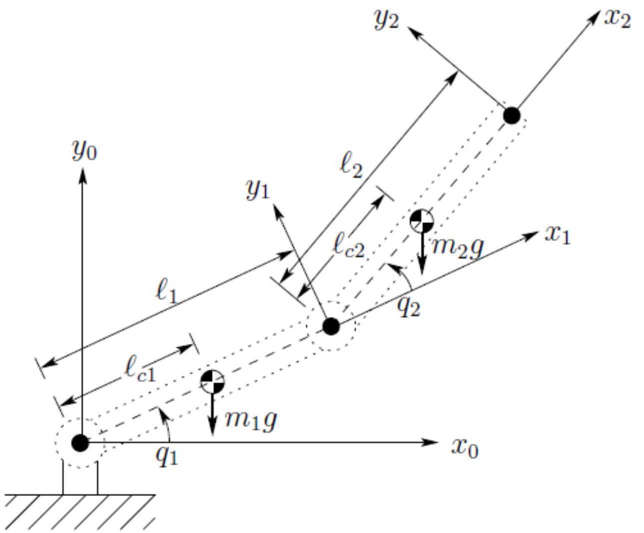
$$\begin{aligned} d_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_1 + I_2 \\ d_{12} &= d_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\ d_{22} &= m_2\ell_{c2}^2 + I_2 \end{aligned}$$

Planar Elbow



$$
\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \\
c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 \ell_1 \ell_{c2} \sin q_2 =: h \\
c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h \\
c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h \\
c_{122} &= c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0
\end{aligned}
$$

Planar Elbow



$$
\begin{aligned}
P_1 &= m_1 g \ell_{c1} \sin q_1 \\
P_2 &= m_2 g(\ell_1 \sin q_1 + \ell_{c2} \sin(q_1 + q_2)) \\
P &= P_1 + P_2 = (m_1 \ell_{c1} + m_2 \ell_1) g \sin q_1 + m_2 \ell_{c2} g \sin(q_1 + q_2)
\end{aligned}
$$

$$
\begin{aligned}
\phi_1 &= \frac{\partial P}{\partial q_1} = (m_1 \ell_{c1} + m_2 \ell_1) g \cos q_1 + m_2 \ell_{c2} g \cos(q_1 + q_2) \\
\phi_2 &= \frac{\partial P}{\partial q_2} = m_2 \ell_{c2} \cos(q_1 + q_2)
\end{aligned}
$$

$$
\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + \phi_1 &= \tau_1 \\
d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + \phi_2 &= \tau_2
\end{aligned}
$$

$$
C = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_2 + h\dot{q}_1 \\ -h\dot{q}_1 & 0 \end{bmatrix}
$$